Classification of some Butler groups of infinite rank *

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Abstract

A new class of torsion-free abelian groups of infinite rank is introduced. Groups of this class are defined as epimorphic images of local acd-groups which admit a special direct decomposition. The main theorems classify the groups from this class up to near-isomorphism for infinite rank groups.

1 Introduction

Our goal is the classification up to near-isomorphism of a class of torsion-free abelian groups which are defined as factor groups of local acd-groups. Recall that an abelian group is completely decomposable (a cd-group) if it is a direct sum of subgroups of the rationals $\mathbb{Q}$ and it is almost completely decomposable (an acd-group) if it is torsion-free abelian of finite rank and has a completely decomposable subgroup of finite index. The class of acd-groups has been studied intensively over the last !three! decades and many structure and non-structure theorems are known. The book by Mader [5] is dedicated to acd-groups and gives a very good account on interesting results derived recently. Even more is known if one assumes that the group is a crq-group, that is a group G which has a !completely decomposable! subgroup A with cyclic quotient G/A. !I am not sure that the next sentence is necessary: This factor group is called a regulating quotient of G, a notion used and discussed in detail in Mader's book [5, Chapter 6].!

In the theory of almost completely decomposable groups and also in general for torsion-free abelian groups of finite rank it has turned out that the concept of near-isomorphism is much more appropriate than the classical notion of

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isomorphism. Recall from Arnold [1, Corollary 12.9(b)] that !*two such groups* X *and* Y *are nearly isomorphic if and only if for each prime* p *there exists a monomorphism* $\eta_p : X \to Y$ *such that* $Y/X\eta_p$ *is a finite group of order relatively prime to* p.! It was therefore reasonable to search for a suitable notion of near-isomorphism for infinite rank abelian groups. This was done in [8] by the first and third authors for a class $\mathcal{H}_{\mathfrak{T}}$ of torsion-free abelian groups that was previously investigated by the first two authors in [9]. A group G in $\mathcal{H}_{\mathfrak{T}}$ is generated by a completely decomposable group C with critical typeset an antichain $\mathfrak{T}$ and some additional elements such that G/C is a special torsion group. A complete classification up to near-isomorphism was obtained in the series of papers [7], [8], [9] for different subclasses of $\mathcal{H}_{\mathfrak{T}}$, namely the class of rigid and the class of almost rigid groups !(the latter has a decomposition into a direct sum of a completely decomposable group and a rigid summand from $\mathcal{H}_{\mathfrak{T}}$ which explains the name "almost rigid groups")!. These classes were obtained by restricting the rank of the homogeneous components of C and the size of $\mathfrak{T}$ as well as the structure of the torsion group $G\backslash C$. Here we will extend those results by considering epimorphic images of rigid and almost rigid groups. Two classification theorems up to near-isomorphism are obtained by carrying over the near-isomorphism classifications for rigid and almost rigid groups. The results can be seen as a generalization of the known results in two ways, namely into the depth and into the width.

Recall that a subgroup of a torsion-free group is pure is the same as to say that the quotient is torsion-free.! The purification of a subgroup U of G is $U_* = \{g \in G : \text{ there is } n \in \mathbb{N} \text{ with } ng \in U\}$. This is also needed in the classical definition of types, see Fuchs [4, p. 85] and note that for a type τ and a group G there is the fully invariant subgroup $G(\tau) = \{g \in G : \mathrm{tp}^G(g) \geq \tau\}$ of G. !If $T \subseteq \hat{\mathfrak{T}}$, with $\hat{\mathfrak{T}}$ the lattice of all types, then $G(\tau)$ naturally generalizes to $G(T) = \sum_{\tau\in T} G(\tau)$.! Following standard definition also $G^*(\tau) = \sum_{\sigma>\tau} G(\sigma)$ and $G^\sharp(\tau)$ is the purification of $G^*(\tau)$ in G. A type τ is critical, a member of $T_{cr}(G)$, if $G(\tau)/G^\sharp(\tau) \neq 0$. !Moreover, a countable torsion-free abelian group G is *block-rigid* if its critical typeset is an antichain. In this case $G(\tau)$ $(\tau \in T_{cr}(G))$ is a homogeneous group, which is called a τ-homogeneous component of G and denoted by G_τ. If all these homogeneous components of a block-rigid group G have rank 1 then G is called a *rigid* group.!

!Throghout the paper we deal with groups of *ring* type having the critical typesets consisting only of idempotent types (which can be represented by characteristics, consisting only of $0'$s and ∞'s, see [5, p. 13], [4, Section 85]). We write $\tau(p) = \infty$ if on the position of a prime p we have ∞.!

For any prime p we denote by $\chi_X^p(x)$ the p-**height** of an element $x \neq 0$ in a group X, which is the p-entry of the height sequence, called the **characteristic** $\chi_X(x)$ of $x \neq 0$ in X, see [4, Section 85].

!If an integer q is divisible by an integer p, we write $p|q$. The symbol $|g|$ denotes the order of a group element $g \in G$, $|G|$ serves as the cardinality of a group G.!

!Imbedding a torsion-free abelian group G in its divisible hull $\mathbb{Q}G = G \otimes \mathbb{Q}$ we denote the group Y by $\frac{G}{q}$ if $qY = G$. Remark that the dimension of the vector space $\mathbb{Q}G$ coincides with $\operatorname{rk} G$, the rank of the group G. This tool will be used for the further linear algebra applications.!

2 Near-isomorphism for infinite rank torsion-free abelian groups

The following class of torsion-free abelian groups was defined in [8]:

Definition 2.1. A torsion-free abelian group X belongs to the class $\mathcal{H}_{\mathfrak{T}}$ if there exists a completely decomposable subgroup $R(X) = \bigoplus_{\tau \in T_{cr}(R(X))} C_\tau^X$ of X such that the following conditions are satisfied:

(1.) $\mathfrak{T} = T_{cr}(R(X))$ is an antichain of idempotent types;

(2.) $R(X)_\tau := C_\tau^X \subseteq_* X$ is a pure and τ-homogeneous completely decomposable subgroup of X for all $\tau \in \mathfrak{T}$;

(3.) $X/R(X) = \bigoplus_{p \in P_X} T_p^X$ for some set of primes P_X and $p^{e_p^X}$-bounded p-groups T_p^X;

(4.) for every $p \in P_X$ the set $\{q \in P_X : [T_p^X] \cap [T_q^X] \neq \varnothing\}$ is finite; here, $[T_p^X]$ is the minimal subset $\mathfrak{T}_p \subseteq \mathfrak{T}$ such that $T_p^X \subseteq ((\bigoplus_{\tau \in \mathfrak{T}_p} C_\tau^X)_* + R(X))/R(X)$.

It is easy to see that the minimal subset $[T_p^X]$ in condition (4.) is uniquely determined for each $p \in P_X$.

A group $X \in \mathcal{H}_{\mathfrak{T}}$ is **almost rigid** if all its homogeneous components C_τ^X are of finite rank, the set $\mathfrak{T}$ is countable and all T_p^X are primary cyclic groups. These groups were introduced and classified up to near-isomorphism in [7]. Observe that condition (4.) implies that each $\tau \in \mathfrak{T}$ is contained in finitely many sets $[T_p^X]$ only. Hence for almost rigid groups this is equivalent to saying that they are finitely presented, see [7, p. 3413 (after Definition 3.2)]. Moreover, we say that a group $X \in \mathcal{H}_{\mathfrak{T}}$ belongs to the class $\mathcal{H}'_{\mathfrak{T}}$ if the rank of $R(X)_\tau$ is finite for all $\tau \in \mathfrak{T}$, see [8, Definition ?]. Note that almost rigid groups belong to the class $\mathcal{H}'_{\mathfrak{T}}$. Finally, following the classical definition by Fuchs [4] we call a group X from the class $\mathcal{H}'_{\mathfrak{T}}$ a **rigid group** if $\mathrm{rk}(R(X)_\tau) = 1$ for each $\tau \in \mathfrak{T}$.

We will need the following

Observation 2.2. Let $X \in \mathcal{H}_{\mathfrak{T}}$.

(1.) If $L \subset R(X)$ is of finite rank then L_*^X is an almost completely decomposable group;

(2.) If X is an almost rigid group and $L \subset R(X)$ is of finite rank then L_*^X is a crq-group;

(3.) $R(X)$ is a uniquely determined !maximal! completely decomposable fully invariant subgroup of X.

By the above observation it is obvious that the groups from the class $\mathcal{H}_{\mathfrak{T}}$ are in fact **local acd-groups** with **regulator** $R(X)$, while the almost rigid groups are contained in the class of **local crq-group**, see [7, Definition 1.1].

Recall from [8, Definition 2.3], [8, Proposition 3.4] the near-isomorphism definition for groups of arbitrary rank.

Definition 2.3. Let X and Y be torsion-free abelian groups. Then X and Y are called **nearly isomorphic**, $X \cong_{nr} Y$ if for every prime p there exist monomorphisms $\eta_p : X \to Y$ and $\mu_p : Y \to X$ such that

(1.) $Y/X\eta_p$ and $X/Y\mu_p$ are torsion;

(2.) $(Y/X\eta_p)_p = 0 = (X/Y\mu_p)_p$;

(3.) for all finite rank pure subgroups $X' \subseteq X$ and $Y' \subseteq Y$ the quotients $(X'\eta_p)_*^Y/X'\eta_p$ and $(Y'\mu_p)_*^X/Y'\mu_p$ are finite.

Clearly, this definition coincides with the classical definition of near-isomorphism for finite rank torsion-free abelian groups (see, for instance, [5, p. 173, Definition 9.1.2]). In fact, !restricting ourselves to the class $\mathcal{H}'_{\mathfrak{T}}$!we are allowed to simplify the near isomorphism notion even more for our purposes!:

Lemma 2.4. *Let $X, Y \in \mathcal{H}'_{\mathfrak{T}}$. Then $G \cong_{nr} H$ if and only if for every prime p there exists a monomorphism $\eta_p : X \to Y$ such that $Y/X\eta_p$ is a torsion group and $(Y/X\eta_p)_p = 0$.*

Proof. It was shown in [8, Lemma 3.3] that for any monomorphisms $\eta_p : X \to Y$ and $\mu_p : Y \to X$ satisfying the condition (3.) of Definition 2.3 holds for the groups X, Y from $\mathcal{H}'_{\mathfrak{T}}$. Moreover, the existence of one-sided monomorphisms $\eta_p : X \to Y$, satisfying conditions (1.) - (2.) of Definition 2.3, is sufficient for groups of this class to be nearly isomorphic by [8, Corollary 4.8].

□

If $|\mathfrak{T}| < \infty$ then the groups in $\mathcal{H}'_{\mathfrak{T}}$ are torsion-free abelian groups of finite rank (more precisely, acd groups), and the given near isomorphism definition coincides for them with the classical definition of near isomorphism. Moreover, it was shown for almost rigid groups in [7, Theorem 4.12 and 5.2] and more generally for countable rank groups from $\mathcal{H}'_{\mathfrak{T}}$ in [9, Theorem 3.4] that they satisfy a theorem !which! extends the classical result by Arnold [1, Corollary 12.9(b)] from finite rank torsion-free abelian groups to groups of countable rank:

Theorem 2.5. *Let X and Y be nearly isomorphic countable groups from the class $\mathcal{H}'_{\mathfrak{T}}$. If $X = X_1 \oplus X_2$ then there exists a decomposition $Y = Y_1 \oplus Y_2$ such that $X_1 \cong_{nr} Y_1$ and $X_2 \cong_{nr} Y_2$.*

3 Rigid groups from the class $\mathcal{H}_{\mathfrak{T}}$ and their epimorphic images

For a rigid group $X \in \mathcal{H}'_{\tau}$ let $R(X)_\tau = \tau a_\tau$ with rank-one rational groups τ containing $\mathbb{Z}$ and let $R(X) = \bigoplus_{\tau \in \mathfrak{T}} \tau a_\tau$ be the regulator of X, a rigid completely decomposable group.

Definition 3.1. Let X be a rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$ and let α_τ ($\tau \in \mathfrak{T}$) be integers such that the following conditions hold:

(1.) $R(X) = \bigoplus_{i\in\lambda} A_i$ with $\mathfrak{T} = \bigcup_{i\in\lambda} T_i$ the disjoint union of finite subsets $T_{cr}(A_i) = T_i \subset \mathfrak{T}$ and $\operatorname{rk} A_i = |T_i| \neq 2$ for any $i \in \lambda$;

(2.) for any $i \in \lambda$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$!for all $\sigma \in T_i$ if $\sigma \neq \tau$!;

(3.) if $i \in \lambda$ and $|T_i| \neq 1$ then $\bigcap_{\tau\neq\sigma,\tau\in T_i} \tau = \mathbb{Z}$ for any $\sigma \in T_i$;

(4.) if $i \in \lambda$ and $|T_i| \neq 1$ then no α_τ $(\tau \in T_i)$ is p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_i$; if $|T_i| = 1$ then $\alpha_\tau = 0$ with $T_i = \{\tau\}$;

(5.) if $i \in \lambda$ and $|T_i| \neq 1$ then $\gcd(\{\alpha_\tau | \, \tau \neq \sigma, \tau \in T_i\}) = 1$ for any $\sigma \in T_i$;

(6.) $|[T_p^X] \cap T_i| \leqslant 1$ for any $p \in P_X$ and $i \in \lambda$;

(7.) if $i \in \lambda$ and $|[T_p^X] \cap T_i| = 1$ for some $p \in P_X$ then α_τ is relatively prime with p for any $\tau \in T_i$;

(8.) if $i \in \lambda$ and $|[T_p^X] \cap T_i| = 1$ then $\tau(p) \neq \infty$ for any $\tau \in T_i$.

The partition $(T_i : \ i \in \lambda)$ and the integers $(\alpha_\tau : \ \tau \in \mathfrak{T})$ are called a **pair of partition and coefficients** for the group X.

Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be a rigid group and $(T_i : \ i \in \lambda)$, $(\alpha_\tau : \ \tau \in \mathfrak{T})$ be a pair of partition and coefficients for X. For simplicity we denote $P = P_X$ and $\gamma_p = e_p^X$.

We put

$$K = \bigoplus_{i\in\lambda} K(A_i) \subset X$$

with $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle$. The group $B = X/K$ is called a rigid **proper local $\mathfrak{B}^{(1)}$asd-group**.

Define

$$\mathcal{A} = \bigoplus_{i\in\lambda} A_i/K(A_i). \tag{1}$$

We see that $\mathcal{A}$ is a **strongly decomposable** rigid group (**sd-group**) as it is a direct sum of strongly indecomposable groups and $\mathfrak{T} = T_{cr}(\mathcal{A})$ is an antichain, [10, Definition 2.1]. Recall that a group is strongly indecomposable if it is not quasi-isomorphic to a decomposable group, see [4]. Moreover, a group is said to be **almost strongly decomposable** (**asd-group**) if it contains a strongly decomposable group as a subgroup of finite index, see [10, Definition 2.3].

Definition 3.2. Let $X \in \mathcal{H}'_{\mathfrak{T}}$ be rigid and $(T_i: \ i \in \lambda)$, $(\alpha_\tau: \ \tau \in \mathfrak{T})$ be a pair of partition and coefficients for X. The canonical epimorphism $\phi: \ X \longmapsto B = X/K$ is called a **regular representation** of the rigid proper local $\mathfrak{B}^{(1)}$asd-group B with the pair $(T_i: \ i \in \lambda)$, $(\alpha_\tau: \ \tau \in \mathfrak{T})$ of partition and coefficients.

By Definition 2.1 and from the fact that $X \subset \mathbb{Q}R(X)$ we derive the so-called **factor primary representation** of X in the following form,

$$X = \sum_{p \in P} X_p \quad \text{with} \quad X_p = X \cap \frac{R(X)}{p^{\gamma_p}} \in \mathcal{H}'_{\mathfrak{T}} \quad \text{and} \quad X_p/R(X) \cong T_p^X. \tag{2}$$

Clearly, $X/R(X) = \bigoplus_{p \in P} X_p/R(X)$. We have the following almost evident

Lemma 3.3. ([9, Lemma 2]) *Let $X = \sum_{p \in P} X_p \in \mathcal{H}'_{\mathfrak{T}}$ be the factor primary representation. Then*
1) X_p is fully invariant in X for any $p \in P$;
2) $X_p \cap X_q = R(X)$ if $p \neq q$.

Remark 3.4. The fully invariant subgroups X_p, $(p \in P)$ play the decisive role in establishing near isomorphism because two groups X, Y from $\mathcal{H}'_{\mathfrak{T}}$ are nearly isomorphic if and only if $X_p \cong_{nr} Y_p$ for each prime p under the natural assumption that $X_p \cong R(X)$ if $p \notin P_X$, see Definition 2.1, [8, Corollary 3.6].

Besides X_p we introduce fully invariant subgroups $X'_p = [T_p^X]_*^{X_p}$ and use them in the following

Remark 3.5. Two groups X, Y from $\mathcal{H}'_{\mathfrak{T}}$ are nearly isomorphic if and only if $R(X) \cong R(Y)$, $P_X = P_Y$ and $X'_p \cong_{nr} Y'_p$ for each prime $p \in P_X = P_Y$.

The factor primary representation $X = \sum_{p \in P} X_p$ implies

$$X/K = B = \sum_{p \in P} B_p \tag{3}$$

with proper local $\mathfrak{B}^{(1)}$asd-groups $B_p = \phi(X_p) = X_p/K$. Moreover, $B/\mathcal{A} = \bigoplus_{p \in P} B_p/\mathcal{A}$ and $p^{\gamma_p}(B_p/\mathcal{A}) = 0$. Evidently, the restriction of ϕ to X_p induces the regular representation $\phi: \ X_p \longmapsto B_p = X_p/K$ of the proper local $\mathfrak{B}^{(1)}$asd-group B_p with the pair $(T_i: \ i \in \lambda)$, $(\alpha_\tau: \ \tau \in \mathfrak{T})$ of partition and coefficients for X_p.

It follows from the factor primary representation (2) of X that $\chi^p_{X_p}(x) = \chi^p_X(x)$ for the p-heights of an element $x \in X_p$.

Lemma 3.6. *Let $\phi : X \longmapsto B = X/K$ be a regular representation of a rigid proper local $\mathfrak{B}^{(1)}$asd-group B with the pair $(T_i : i \in \lambda)$, $(\alpha_\tau : \tau \in \mathfrak{T})$ of partition and coefficients for X. Let $R(X) = \bigoplus_{\tau \in T_{cr}(X)} \tau a_\tau$. Then for any prime $p \in P$ and $\tau \in \mathfrak{T}$ the following holds:*

$$0 = \chi^p_{R(X)}(a_\tau) = \chi^p_X(a_\tau) = \chi^p_{X_p}(a_\tau) = \chi^p_{\mathcal{A}}(\phi(a_\tau)) = \chi^p_{B_p}(\phi(a_\tau)) = \chi^p_B(\phi(a_\tau)).$$

Proof. Fix a prime $p \in P$. Only the last three of the required equalities need to be verified because the others concern groups of $\mathcal{H}'_{\mathfrak{T}}$ and, thus, follow from Definition 2.1 and (2). By construction, $\chi^p_{\mathcal{A}}(\phi(a_\tau)) = \chi^p_{R(X)}(a_\tau)$ because the restriction of ϕ to any A_i ($i \in \lambda$) defines a proper $\mathfrak{B}^{(1)}$-group $A_i/K(A_i)$, which is cotrimmed, that is the image of each τa_τ is pure in $A_i/K(A_i)$ under the canonical mapping $A_i \longrightarrow A_i/K(A_i)$ if $\tau \in T_i$, see [10, Definition 2.1].

For the next equality assume that $\chi^p_X(a_\tau) < \chi^p_{B_p}(\phi(a_\tau))$. Then there exists a preimage of $\phi(a_\tau)$ in X, which is divisible by p. However, this is impossible because neither $\alpha_\tau a_\tau$, a preimage of $\phi(\alpha_\tau a_\tau)$, nor $\sum_{\sigma \in T_i, \sigma \neq \tau} \alpha_\sigma a_\sigma$ is divisible by p if $\tau \in T_i$, see Definition 3.1 (3.) – (8.).

The last equality follows from (3). The proof is complete. □

Corollary 3.7. Let $\phi : X \longmapsto B = X/K$ be a regular representation of a rigid proper local $\mathfrak{B}^{(1)}$asd-group B. Then $X'_p \cong X'_p\phi$ are fully invariant subgroups of X and B respectively for any prime $p \in P$ (see Definition 3.1(6.)).

Lemma 3.6 implies that the groups $A_i/K(A_i)$ are pure in B which is the reason to call $\mathcal{A} = \bigoplus_{i \in \lambda} A_i/K(A_i)$ the **regulator** of B and to write $\mathcal{A} = R(B)$.

On the basis of the above Lemma using $0 = \chi^p_{R(X)}(a_\tau) = \chi^p_{\mathcal{A}}(\phi(a_\tau))$, for each $p \in P$ we construct the following commutative diagram ($\times p^{\gamma_p}$ means multiplication by p^{γ_p}),

$$\begin{array}{ccccc} X_p & \overset{\times p^{\gamma_p}}{\longrightarrow} & R(X) & \overset{-}{\longrightarrow} & \overline{R(X)} = R(X)/p^{\gamma_p}R(X) \\ \downarrow \phi & & \downarrow \phi|_{R(X)} & & \downarrow \overline{\phi} \\ B_p & \overset{\times p^{\gamma_p}}{\longrightarrow} & \mathcal{A} & \overset{-}{\longrightarrow} & \overline{\mathcal{A}} = \mathcal{A}/p^{\gamma_p}\mathcal{A} \end{array} \tag{4}$$

with the natural epimorphism $\overline{\phi} : \overline{R(X)} \longrightarrow \overline{\mathcal{A}} \cong \overline{R(X)}/\overline{K}$ defined on $\overline{R(X)} = \bigoplus_{\tau \in T} \langle \overline{a_\tau} \rangle$ by $\overline{K} = \bigoplus_{i \in \lambda} \overline{K(A_i)}$ and $\overline{K(A_i)} = \langle \sum_{\tau \in T_i} \alpha_\tau \overline{a_\tau} \rangle$, see Definition 3.1.

Denote

$$B_i = A_i/K(A_i) \quad \text{for} \quad i \in \lambda. \tag{5}$$

Remark 3.8. The name "proper local $\mathfrak{B}^{(1)}$asd-group" for the group $B = X/K$ reflects the fact that $\mathcal{A} = R(X)/K$ is a strongly decomposable group with strongly indecomposable summands $B_i = A_i/K(A_i)$, and the quotient of B over $\mathcal{A}$ is torsion and equal to $B/\mathcal{A} \cong (X/K)/(R(X)/K) \cong X/R(X)$ which is isomorphic to $\bigoplus_{p \in P} T_p^X$, see Definition 2.1.

Groups of the form B_i were called proper $\mathfrak{B}^{(1)}$-groups in [10, Definition 2.1] and will be considered in the next section. We do not need to quote here a definition of a proper $\mathfrak{B}^{(1)}$-group B_i because it is equivalent to the conditions (2.)–(5.) of Definition 3.1. In fact, each B_i is defined up to isomorphism by the set of pairwise incomparable idempotent types T_i with $|T_i| \neq 2$ and the set of coefficients α_τ $(\tau \in T_i)$. Moreover, $\operatorname{End} B_i \cong \mathbb{Z}$, see [10, Remark 2.2].

Take a pure subgroup $B' = (\bigoplus_{i \in I} B_i)_*^B$ of B with finite subset I of λ. By Definition 2.1 (4.) and Lemma 3.6 we conclude that it is an almost strongly decomposable group as $A' = (\bigoplus_{i \in I} A_i)_*^X$ is an almost completely decomposable group with $A'/R(A') \cong (A'/\bigoplus_{i \in I} K(A_i))/(R(A')/\bigoplus_{i \in I} K(A_i)) \cong B'/R(B')$.

Recall from [8, Proposition 3.2] that if X and Y are nearly isomorphic rigid groups from the class $\mathcal{H}'_{\mathfrak{T}}$ then $R(X)$ and $R(Y)$ are isomorphic and can be identified.

Lemma 3.9. *Let X, Y be groups from the class $\mathcal{H}'_{\mathfrak{T}}$ with regulator R and $\Phi \in \operatorname{Mon}(X, Y)$. If $(R/R\Phi)_p = 0$ then $(Y/X\Phi)_p \cong (Y_{(p)}/X_{(p)}\Phi)_p$.*

Proof. We have $(Y/X\Phi)_p \cong (Y/R\Phi)_p/(X\Phi/R\Phi)_p$ and $(Y/R)_p = Y_{(p)}/R \cong (Y_{(p)}/R\Phi)/(R/R\Phi)$.

From $(R/R\Phi)_p = 0$ it follows that $Y_{(p)}/R = (Y_{(p)}/R)_p \cong (Y_{(p)}/R\Phi)_p$. Similarly, $Y_{(p)}/R \cong (Y/R)_p \cong (Y/R\Phi)_p/(R/R\Phi)_p \cong (Y/R\Phi)_p$. Then $(Y/R\Phi)_p \cong (Y_{(p)}/R\Phi)_p$.

Furthermore, $(X\Phi/R\Phi)_p \cong (X/R)_p \cong X_{(p)}/R \cong X_{(p)}\Phi/R\Phi = (X_{(p)}\Phi/R\Phi)_p$ as Φ is injective. Therefore, $(Y_{(p)}/X_{(p)}\Phi)_p \cong (Y_{(p)}/R\Phi)_p/(X_{(p)}\Phi/R\Phi)_p \cong (Y/R\Phi)_p/(X\Phi/R\Phi)_p \cong (Y/X\Phi)_p$ as required.

□

We are ready to state our main theorem that classifies rigid proper local $\mathfrak{B}^{(1)}$asd-groups up to near-isomorphism.

Theorem 3.10. *Let B and C be rigid proper local $\mathfrak{B}^{(1)}$asd-groups and $\phi : X \longmapsto B$, $\psi : Y \longmapsto C$ be their regular representations. Then $B \cong_{nr} C$ if* *and only if $X \cong_{nr} Y$ and the pairs of partition and coefficients for X and Y coincide.*

Proof. Let B and C be nearly isomorphic groups. Let the monomorphisms $\eta_p : B \longrightarrow C$ and $\mu_p : C \longrightarrow B$ satisfy the required conditions for all primes p, that is $C/B\eta_p$ and $B/C\mu_p$ are torsion groups with zero p-components. Clearly, $T_{cr}(B) = T_{cr}(C)$ and the restrictions of these monomorphisms to the regulators also satisfy the property $(R(C)/R(B)\eta_p)_p = (R(B)/R(X)\mu_p)_p = 0$ because $(R(C)_\tau/R(B)_\tau\eta_p)_p$ and $(R(B)_\tau/R(C)_\tau\mu_p)_p$ are finite subgroups of $C/B\eta_p$ and $B/C\mu_p$ respectively for any $\tau \in T_{cr}(B) = T_{cr}(C)$. Then $R(C) \cong_{nr} R(B)$ by Definition 2.3 and it follows that $R(C) \cong R(B)$ because the regulators are rigid groups which are direct sums of Butler groups with endomorphism rings $\mathbb{Z}$ and near isomorphism for them coincides with isomorphism, see [1], [2, p. 26]. This implies $R(X) \cong R(Y)$ and the regular representations of B and C have the same partition of $\mathfrak{T} = T_{cr}(B) = T_{cr}(C) = T_{cr}(X) = T_{cr}(Y) = \bigcup_{i\in\lambda} T_i$ and the same coefficients α_τ, $\tau \in \mathfrak{T}$, see Definition 3.1.

Without loss of generality assume $\tilde{R} = R(C) = R(B)$, then any monomorphism $\mu_p : C \longrightarrow B$ can be considered as a monomorphism of $\tilde{R}$, and μ_p uniquely extends to the divisible hull $\mathbb{Q}\tilde{R}$ of the regulator. It is also a monomorphism of its p-divisible hull $\mathbb{Q}_{(p)}\tilde{R}$ with $\mathbb{Q}_{(p)} = \langle \frac{1}{p^k} : \ k \in \mathbb{N} \rangle$ which leads to the following: $C_p\mu_p \subset B_p$ and $(B_p/C_p\mu_p)_p = 0$ for each p, see (3). Denote $B'_p = X'_p\phi$ and $C'_p = Y'_p\phi$, clearly, $B'_p \cong_{nr} C'_p$. By construction, $B'_p \cong X'_p$ and $C'_p \cong Y'_p$. Then $X'_p \cong_{nr} Y'_p$ and also $X_p \cong_{nr} Y_p$ for each prime p, hence $X \cong_{nr} Y$ by Remark 3.4.

Conversely, let $R = R(X) = R(Y) = \bigoplus_{i\in\lambda}\bigoplus_{\tau\in T_i} \tau a_\tau$ with $X(\tau) = Y(\tau) = R_\tau = \tau a_\tau$. Denote $K_1 = \operatorname{Ker}\phi$ and $K_2 = \operatorname{Ker}\psi$ and assume that $K_1 = K_2 = \bigoplus_{i=1,\dots,t} \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle$.

!From $X \cong_{nr} Y$ we have $X'_p \cong_{nr} Y'_p$ for any prime p, see Remark 3.5. Then there exist monomorphisms $\chi_p : X'_p \longmapsto Y'_p$ such that the orders of elements from $Y'_p/X'_p\chi_p$ are not divisible by p.! Denote $P = P_X = P_Y$, $T_p = T_p^X = T_p^Y$ and $[T_p] = [T_p^X] = [T_p^Y]$. Fix $p \in P$ and let $a_\tau\chi_p = \alpha_\tau a_\tau$ with $\alpha_\tau \in \tau$ ($\tau \in [T_p]$). Take integers α'_τ such that $\alpha'_\tau - \alpha_\tau \in p^{\gamma_p}\tau$ and put $\tilde{p} = \prod_{q\in P: q\neq p,\ [T_p]\cap[T_q]\neq\varnothing} q^{\gamma_q}$ (the product is finite by Definition 2.1 (4.)). Construct a monomorphism $\tilde{\chi}_p : X'_p \longmapsto Y'_p$ such that $a_\tau\tilde{\chi}_p = \beta_\tau a_\tau$ with integers $\beta_\tau = \tilde{p}\alpha'_\tau$ ($\tau \in [T_p]$). Put $\beta_\tau = \prod_{q\in P:\tau\in[T_q]} q^{\gamma_q}$ if $\tau \notin [T_p]$ and define $p_\tau = \prod_{q\in P:\tau\in[T_q]} q^{\gamma_q}$ for all $\tau \in \mathfrak{T}$ (note that $\beta_\tau = p_\tau$ if $\tau \notin [T_p]$). By the classical Chinese Remainder Theorem there exist integers $\beta_i \equiv \beta_\tau \pmod{p_\tau}$ if $\tau \in T_i$ for each $i \in \lambda$, because p_τ and p_σ from the same T_i are relatively prime if $\tau \neq \sigma$ by Definition 3.1 (6.).

Construct a monomorphism $\eta_p : R \longmapsto R$ defined by $a_\tau \eta_p = \beta_i a_\tau$ for $\tau \in T_i$, $i \in \lambda$. It can be extended to a monomorphism $\eta_p : X \longmapsto Y$ with $\eta_p : X'_p \longmapsto Y'_p$ and $\eta_q : X'_p \longmapsto R$ for all $q \neq p$. By construction, the orders of elements of torsion groups $R/R\eta_p$ and $Y_p/X_p\eta_p$ are not divisible by p, then the same is true for $Y/X\eta_p$ by Lemma 3.9.

If $p \notin P$, set $\beta_i = \prod_{q \in P:\ T_i \cap [T_q] \neq \varnothing} q^{\gamma_q}$ and define a monomorphism $\eta_p : X \longmapsto Y$ again by $a_\tau \eta_p = \beta_i a_\tau$ for $\tau \in T_i$, $i \in \lambda$ (in fact $\eta_p : X \longmapsto R$).

For any prime p denote $Y' = X\eta'_p \cong X$. Then the representation $\psi : Y \longmapsto C$ implies the representation $\psi|_{Y'} : Y' \longmapsto Y'\psi$ with the same coefficients as ϕ and ψ. Therefore, $Y'\psi \subset Y\psi$. Moreover, $Y\psi/Y'\psi$ and Y/Y' are torsion groups having the same p-components because every $Y(\tau)\psi$ is pure in $Y\psi$ by Lemma 3.6.

Since $B \cong Y'\psi$ and $C = Y\psi$, we have a monomorphism $\eta_p : B \longmapsto C$ defined by $a_\tau \psi \eta_p = \beta_i a_\tau \psi$ $(\tau \in \mathfrak{T})$ such that C/B is a torsion group with zero p-component. By symmetry, from $X \cong_{nr} Y$ we can obtain embedding $\mu_p : C \longmapsto B$ with torsion B/C whose p-component is also zero.

The near-isomorphism of B and C has been established because condition (3.) of Definition 2.3 holds for η_p and μ_p again by Lemma 3.6.

□

Taking into account that for a rigid proper local $\mathfrak{B}^{(1)}$asd-group its regular representation $\phi : X \longmapsto B$ satisfies $X'_p \cong \phi(X'_p)$ (assuming $X'_p = 0$ if $p \notin P$) we give the following intrinsic equivalent definition of this class of groups without reference to the map ϕ, see Definitions 3.1 and 2.1. This will enable us to reformulate Theorem 3.10 without referring to regular representations.

Definition-Lemma 3.11. *Let B be a group with an antichain $\mathfrak{T} = T_{cr}(B) = \bigcup_{i\in\lambda} T_i$ for pair-wise disjoint sets of idempotent types T_i and $\mathcal{A} \cong \bigoplus_{i\in\lambda} B(A_i)$ be its strongly decomposable subgroup with $B(A_i) = A_i/K(A_i)$ defined by a rigid completely decomposable finite rank group $A_i = \bigoplus_{\tau\in T_i} \tau a_\tau$ and $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle$ $(\alpha_\tau \in \mathbb{Z},\ \tau \in T_i)$. Suppose that $B/\mathcal{A} \cong \bigoplus_{p\in P} T_p$ for some set of primes P and p^{γ_p}-bounded p-groups T_p $(\gamma_p \in \mathbb{N})$.*

Let the following conditions hold:

(1.) for any $i \in \lambda$ and $\tau \in T_i$ the group $\tau a_\tau + K(A_i)$ is pure in B and $\operatorname{rk} A_i = |T_i| \neq 2$;

(2.) there exist minimal sets $[T_p] \subset \mathfrak{T}$, such that the factor-groups $(B \cap \frac{\bigoplus_{\tau\in[T_p]} \mathcal{A}(\tau)}{p^{\gamma_p}} / \bigoplus_{\tau\in[T_p]} \mathcal{A}(\tau))$ are isomorphic to corresponding T_p and $|[T_p] \cap T_i| \leqslant 1$ for any $p \in P$ and any $i \in \lambda$;

(3.) for every $p \in P$ the set $\{q \in P : [T_p] \cap [T_q] \neq \varnothing\}$ is finite;

(4.) for any $i \in \lambda$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$, $\sigma \in T_i$;

(5.) if $i \in \lambda$ and $|T_i| \neq 1$ then $\bigcap_{\tau\neq\sigma,\tau\in T_i} \tau = \mathbb{Z}$ for any $\sigma \in T_i$;

(6.) if $i \in \lambda$ and $|T_i| \neq 1$ then each α_τ, $\tau \in T_i$, is not p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_i$, if $|T_i| = 1$ then $\alpha_\tau = 0$ with $T_i = \{\tau\}$;

(7.) if $i \in \lambda$ and $|T_i| \neq 1$ then $\gcd(\{\alpha_\tau | \tau \neq \sigma, \tau \in T_i\}) = 1$ for any $\sigma \in T_i$;

(8.) if $i \in \lambda$ and $|[T_p^X] \cap T_i| = 1$ for some $p \in P_X$ then each α_τ is relatively prime to p for any $\tau \in T_i$;

(9.) if $i \in \lambda$ and $|[T_p^X] \cap T_i| = 1$ then $\tau(p) \neq \infty$ for any $\tau \in T_i$.

Clearly, $R(B) = \mathcal{A}$ is the regulator of B in the sense of Definition 3.1. Denote $B'_p = \phi(X'_p) = B \cap \frac{\bigoplus_{\tau\in[T_p]} \mathcal{A}(\tau)}{p^{\gamma_p}}$ and from Theorem 3.10 and Remark 3.5 we immediately obtain

Theorem 3.12. *Let B and C be rigid proper local $\mathfrak{B}^{(1)}$asd-groups. Then $B \cong_{nr} C$ if and only if their regulators $R(B)$ and $R(C)$ are isomorphic and $B'_p \cong_{nr} C'_p$ for each prime p.*

4 Groups of the form $G^m[A]$

In this section we are interested in direct sums of groups of the form $K_i = A_i/K(A_i)$ $(i \in \lambda)$ for a rigid group A_i . This is a preparation for the theorem of Section 5 about the classification of proper $\mathfrak{B}^{(1)}$alr-groups. We have a first

Definition 4.1. ([10, Definition 2.1], Definition 3.1, (2.) – (5.))

Let $A = \bigoplus_{\tau \in T_{cr}(A)} \tau a_\tau$ be a rigid completely decomposable group of finite rank $k = |T_{cr}(A)| \geqslant 3$ and $\{\alpha_\tau : \tau \in T_{cr}(A)\}$ be integers such that the following hold:

(1.) $T_{cr}(A)$ is an antichain of idempotent types;

(2.) for any $\tau \in T_{cr}(A)$ there is a prime p with $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$;

(3.) $\bigcap_{\tau \in T_{cr}(A),\ \tau \neq \sigma} \tau = \mathbb{Z}$ for any $\sigma \in T_{cr}(A)$;

(4.) each α_τ is not p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_{cr}(A)$;

(5.) $\gcd(\{\alpha_\tau | \, \tau \neq \sigma, \tau \in T_{cr}(A)\}) = 1$ for any $\sigma \in T_{cr}(A)$.

Then

$$G[A] = \sum_{\tau \in T_{cr}(A)} \tau a_\tau \quad \text{with the relation} \quad \sum_{\tau \in T_{cr}(A)} \alpha_\tau a_\tau = 0 \tag{6}$$

is called a **proper $\mathfrak{B}^{(1)}$-group**.

Note that proper $\mathfrak{B}^{(1)}$-groups are contained in the class of proper local $\mathfrak{B}^{(1)}$asd-groups and actually form the regulators of the latter.

Let

$$\phi \ : \ A \longmapsto G[A]$$

denote the natural epimorphism. Recall that a proper $\mathfrak{B}^{(1)}$-group $G[A]$ is cotrimmed (which means that the image of each τa_τ is pure in $G[A]$, see [2, p.19]) and strongly indecomposable, see [4, Section 92], [3, Theorem 3.3 and Corollary 3.5].

From Definition 4.1 we have that for any $\sigma \in T_{cr}(A)$ the elements $\{\alpha_\tau a_\tau : \tau \neq \sigma,\ \tau \in T_{cr}(A)\}$ have no non-trivial common divisors in A. Hence

$$A(\tau) \cong G[A](\tau) \cong \tau a_\tau \quad \text{for all} \quad \tau \in T_{cr}(A). \tag{7}$$

Remark 4.2. It was shown in [3, Lemma 5.1] that any two strongly indecomposable groups of the forms $G[A]$ and $G[A']$ are isomorphic if and only if the completely decomposable groups A and A' are isomorphic and the coefficients α_τ in the definitions of $G[A]$ and $G[A']$ coincide for any $\tau \in T_{cr}(A) = T_{cr}(A')$, see Definition 4.1.

According to Definition 4.1 we conclude that $G[A] = A/K(A)$, where

$$K(A) = \langle \sum_{\tau \in T_{cr}(A)} \alpha_\tau a_\tau \rangle \cong \mathbb{Z} \tag{8}$$

is the rank-one subgroup defined by the nonzero integers α_τ ($\tau \in T_{cr}(A)$). We need to examine a direct sum of finitely many groups isomorphic to $G[A]$ for a particular i. To simplify our notation let us fix i and denote $A = A_i$, $T = T_i = T_{cr}(A)$ and $G[A] \cong B_i = A/K(A_i)$ with fixed $(\alpha_\tau : \tau \in T)$. Let

$$G^m[A] = \bigoplus_{n=1}^{m} G[A, n] \tag{9}$$

be the direct sum of m copies of $G[A]$, $G[A, n] \cong G[A]$ for any $n = 1, \ldots, m$ and $K = K^m(A) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ be the naturally defined subgroup of rank m of

$$A^m \cong A \oplus \ldots \oplus A$$

generated by the element $\sum_{\tau \in T} \alpha_\tau a_\tau$ in each summand isomorphic to A for a fixed decomposition of A^m.

Lemma 4.3. $\operatorname{End} G^m[A]$ *is isomorphic to the* $(m \times m)$*-matrix ring* $\operatorname{Mat}_{(m\times m)}(\mathbb{Z})$ *over the ring of integers.*

Proof. It was shown in [3, Corollary 3.5] that $\operatorname{End} G[A] \cong \bigcap_{\tau \in T} \tau = \mathbb{Z}$. It means that any endomorphism of $G[A]$ acts on each a_τ ($\tau \in T_{cr}(A)$) as multiplication by the same integer. Then the endomorphism ring of $G^m[A]$ is isomorphic to that of the free module of rank m over the principal ideal domain $\mathbb{Z}$ and can be represented by the matrix ring $\operatorname{Mat}_{(m\times m)}(\mathbb{Z})$. □

Remark 4.4. Clearly, the set of monomorphisms of $G^m[A]$ is given by

$$\operatorname{Mon} G^m[A] = \{D \in \operatorname{Mat}_{(m\times m)}(\mathbb{Z}) :\ \det D_m \neq 0\}$$

and the automorphism group is given by

$$\operatorname{Aut} G^m[A] = \{D \in \operatorname{Mat}_{(m\times m)}(\mathbb{Z}) :\ |\det D_m| = 1\}.$$

It follows from the classical Krull-Schmidt Theorem for free modules of rank m over $\mathbb{Z}$, that any decomposition of $G^m[A]$ into a direct sum of indecomposable groups looks like the original one, that is any indecomposable summand of $G^m[A]$ is isomorphic to $G[A]$ and the number of them equals m.

We return to the group A^m with $T = T_{cr}(A)$ and define a special class of its automorphisms.

Definition 4.5. Any set of elements $(A^m)_K = \{a_\tau^n \in A^m | \, \tau \in T, \; n = 1, \ldots, m\}$ will be called a K-**basis** of A^m if the following hold:

1. $A^m(\tau) = \bigoplus_{n \leqslant m} \tau a_\tau^n$ with $\tau \in T$ $(\mathbb{Z} \subset \tau \subset \mathbb{Q})$;
2. $K = \langle \{ \sum_{\tau \in T} \alpha_\tau a_\tau^n | \; n = 1, \ldots, m \} \rangle \cong K^m(A)$.

Definition 4.6. Automorphisms $\Psi \in \operatorname{Aut} A^m$, which act on each fully invariant subgroup $A^m(\tau)$ as multiplication by the same matrix $D \quad (\det D = 1)$ with respect to a K-basis a_τ^n, will be called K-**basic** automorphisms of A^m. They form a subgroup $\operatorname{Aut}^K A^m$ of $\operatorname{Aut} A^m$.

Remark 4.7. The name K-**basic automorphisms** of A^m is explained by the fact that the subgroup K is invariant with respect to these automorphisms. Furthermore K can be considered as a free module of rank m over $\mathbb{Z}$ and satisfies the Krull-Schmidt Theorem too. The elements of $\operatorname{Aut}^K A^m$ naturally induce all the automorphisms of $G = A^m/K$.

Any decomposition $G = G_1 \oplus G_2$ implies the corresponding decomposition of each fully invariant subgroup $G(\tau)$, which derives from a decomposition of its pre-image in A^m, because $G(\tau) \cong A^m(\tau)$ for any $\tau \in T$. This leads to the decompositions $A^m = \bigoplus_{\tau \in T} A^m(\tau) = A_1 \oplus A_2$ and $K = K_1 \oplus K_2$!with! $G_1 = A_1/K_1$ and $G_2 = A_2/K_2$.

If we remove the restriction on $\det D$ in Definition 4.6, we obtain the ring $\operatorname{End}^K A^m$ of K-basic endomorphisms of A^m, which are naturally restricted to endomorphisms of K. Evidently $\operatorname{Aut}^K A^m$ and $\operatorname{End}^K A^m$ are independent of the choice of the coefficients α_τ in the definition of K.

5 Almost rigid groups and their epimorphic images

Now we introduce a new class of torsion-free abelian groups which are epimorphic images of almost rigid groups. Recall that a group $X \in \mathcal{H}'_{\mathfrak{T}}$ is an almost rigid group if all its homogeneous components C^X_τ are of finite rank, the set $\mathfrak{T}$ is at most countable and all T^X_p are p-primary cyclic groups, see Definition 2.1 and [7].

Definition 5.1. Let X be an almost rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$ and $(\alpha_\tau : \tau \in \mathfrak{T})$ be integers such that the following conditions hold:

(1.) $R(X) = \bigoplus_{i\in\omega} A_i^{l_i}$ $(l_i \in \mathbb{N})$ with $\mathfrak{T} = \bigcup_{i\in\omega} T_i$ the disjoint union of finite subsets $T_{cr}(A_i) = T_i \subset \mathfrak{T}$ and $\operatorname{rk} A_i = |T_i| \neq 2$ for any $i \in \omega$;

(2.) for any $i \in \omega$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$, $\sigma \in T_i$;

(3.) if $i \in \omega$ and $|T_i| \neq 1$ then $\bigcap_{\tau\neq\sigma,\tau\in T_i} \tau = \mathbb{Z}$ for any $\sigma \in T_i$;

(4.) if $i \in \omega$ and $|T_i| \neq 1$ then each α_τ $(\tau \in T_i)$ is not p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_i$; if $|T_i| = 1$ then $\alpha_\tau = 0$ with $T_i = \{\tau\}$;

(5.) if $i \in \omega$ and $|T_i| \neq 1$ then $\gcd(\{\alpha_\tau | \, \tau \neq \sigma, \tau \in T_i\}) = 1$ for any $\sigma \in T_i$;

(6.) $|[T^X_p] \cap T_i| \leqslant 1$ for any $p \in P_X$ and $i \in \omega$;

(7.) if $i \in \omega$ and $|[T^X_p] \cap T_i| = 1$ for some $p \in P_X$ then each α_τ is relatively prime to p for any $\tau \in T_i$;

(8.) if $i \in \omega$ and $|[T^X_p] \cap T_i| = 1$ then $\tau(p) \neq \infty$ for any $\tau \in T_i$.

Let $K = \bigoplus_{i\in\omega} K^{l_i}(A_i) \subset X$ with $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle \subset A_i$.

The partition $(T_i : \; i \in \lambda)$ and two sets of integers $(\alpha_\tau : \; \tau \in \mathfrak{T})$, $(l_i : \; i \in \omega)$ will be called a **triple of partition, coefficients and parameters** for the group X.

The group $B = X/K$ will be called a **proper $\mathfrak{B}^{(1)}$alr-group**.

Obviously, a proper $\mathfrak{B}^{(1)}$alr-group $B = X/K$ is a proper local $\mathfrak{B}^{(1)}$asd-group if X is rigid (see Definition 3.1).

Define

$$\mathcal{A} = R(X)/K = \bigoplus_{i\in\omega} A_i^{l_i}/K^{l_i}(A_i) = \bigoplus_{i\in\omega} (A_i/K(A_i))^{l_i}. \tag{10}$$

Since $(A_i/K(A_i))^{l_i}$ is a group of the form $G^{l_i}[A_i]$, see (9), we extend Definitions 4.5, 4.6 to the preimage of a proper $\mathfrak{B}^{(1)}$alr-group !using Definition 5.1.!

Definition 5.2. Let X be an almost rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$. Any set of elements $(R(X))_K = \bigcup_{i\in\omega}(A_i^{l_i})_{K_i}$ with $K_i = K^{l_i}(A_i) = K \cap X(T_i)$ will be called a K-**basis** of X.

Furthermore, $\mathrm{Aut}^K X$ will denote the group of K-**basic automorphisms** of X, if it consists of all $\Psi \in \mathrm{Aut}\, X$, for which the restriction to every $R(X)(T_i) \cong X(T_i) \cong A_i^{l_i}$ coincides with a K_i-basic automorphism of $X(T_i)$, that is $\Psi\mid_{X(T_i)} = \Psi_i$ with $\Psi_i \in \mathrm{Aut}^{K_i}(A_i^{l_i})$, $i \in \omega$.

Definition 5.3. Let X be an almost rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$ and !let! $(T_i : \ i \in \lambda)$, $(\alpha_\tau : \ \tau \in \mathfrak{T})$ and $(l_i : \ i \in \omega)$ be a triple of partition, coefficients and parameters for X. The canonical epimorphism $\phi : \ X \longmapsto B = X/K$ will be called a **regular representation** of a proper $\mathfrak{B}^{(1)}$alr-group B.

On the basis of the factor primary representation (2) of arbitrary groups from the class $\mathcal{H}'_{\mathfrak{T}}$ we define finite rank fully invariant subgroups $X'_p = [T_p^X]_*^{X_p}$ of an almost rigid group X, which satisfy

$$B'_p = X'_p\phi \cong X'_p. \tag{11}$$

Clearly, B'_p and X'_p are block-rigid crq-groups with primary regulator quotient. Denote $P = P_X$ and $\gamma_p = e_p^X$. From [7, (4,5)] we recall that there are elements

$$u_p \in X \text{ such that } \bigoplus_{p\in P}\langle u_p + R(X)\rangle = X/R(X) \text{ with } |u_p + R(X)| = p^{\gamma_p} \tag{12}$$

and we write

$$p^{\gamma_p}u_p = \sum_{\tau\in\mathfrak{T}} u_{\tau p} \text{ with } u_{\tau p} \in R(X)_\tau. \tag{13}$$

For each $p \in P$ we have a canonical epimorphism $\overline{} : \ R(X) \longrightarrow R(X)/p^{\gamma_p}R(X)$, hence $\overline{p^{\gamma_p}u_p} = \sum_{\tau\in\mathfrak{T}}\overline{u_{\tau p}}$, $\overline{u_{\tau p}} \in R(X)_\tau/p^{\gamma_p}R(X)_\tau$. We use this to define the invariants of an almost rigid group X.

Definition 5.4. ([7, (Definition 3.2)]) If X is an almost rigid group as in Definition 5.1 and $\overline{} : \ R(X) \longrightarrow R(X)/p^{\gamma_p}R(X)$, then we define

$$m_\tau(X) = \prod_{p\in P}|\overline{u_{\tau p}}| \ \text{ for } \ \tau \in \mathfrak{T}.$$

Remark 5.5. It follows from Definition 2.1 (4.) that $m_\tau(X)$ $(\tau \in \mathfrak{T})$ are finite because $|\overline{u_{\tau p}}| = 1$ for almost all $p \in P$. Note also that all the sets $[T_p^X]$ $(p \in P)$ are finite for an almost rigid group $X \in \mathcal{H}'_{\mathfrak{T}}$. Recall from [7] that the regulator $R(X)$ and the numbers $m_\tau(X)$ $(\tau \in \mathfrak{T})$ define a group X in the class of all almost rigid groups up to near-isomorphism. This is the reason to call $m_\tau(X)$ the **near-isomorphism invariants** of the group X. Clearly, $\tau(X) \neq \infty$ for any prime $p | m_\tau(X)$.

We use the near-isomorphism invariants to give an equivalent definition of a $\mathfrak{B}^{(1)}$alr-group (see Definition 5.1):

Definition-Lemma 5.6. *Let X be an almost rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$ and $(\alpha_\tau : \tau \in \mathfrak{T})$ be integers such that the following conditions hold:*

(1.) $R(X) = \bigoplus_{i\in\omega} A_i^{l_i}$ $(l_i \in \mathbb{N})$ with $\mathfrak{T} = \bigcup_{i\in\omega} T_i$ the disjoint union of finite subsets $T_{cr}(A_i) = T_i \subset \mathfrak{T}$ and $\operatorname{rk} A_i = |T_i| \neq 2$ for any $i \in \omega$;

(2.) for any $i \in \omega$ and any $\tau \in T_i$ there exists a prime p such that $\tau(p) = \infty$ and $\sigma(p) \neq \infty$ for all $\sigma \neq \tau$, $\sigma \in T_i$;

(3.) if $i \in \omega$ and $|T_i| \neq 1$ then $\bigcap_{\tau\neq\sigma,\tau\in T_i} \tau = \mathbb{Z}$ for any $\sigma \in T_i$;

(4.) if $i \in \omega$ and $|T_i| \neq 1$ then each α_τ $(\tau \in T_i)$ is not p-divisible if $\sigma(p) = \infty$ for some $\sigma \in T_i$; if $|T_i| = 1$ then $\alpha_\tau = 0$ with $T_i = \{\tau\}$;

(5.) if $i \in \omega$ and $|T_i| \neq 1$ then $\gcd(\{\alpha_\tau | \tau \neq \sigma, \tau \in T_i\}) = 1$ for any $\sigma \in T_i$;

(6.) if there exists $i \in \omega$ with $\tau \in T_i$ and $\sigma \in T_i$ then $\gcd(m_\tau(X), m_\sigma(X)) = 1$;

(7.) if there exists $i \in \omega$ with $\tau \in T_i$ and $\sigma \in T_i$ then α_τ is relatively prime to $m_\sigma(X)$;

(8.) if there exists i with $\tau \in T_i$ and $\sigma \in T_i$ then $\tau(p) \neq \infty$ for any prime divisor p of $m_\sigma(X)$.

Let $K = \bigoplus_{i\in\omega} K^{l_i}(A_i) \subset X$ with $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle \subset A_i$.

The partition $(T_i : i \in \lambda)$ and two sets of integers $(\alpha_\tau : \tau \in \mathfrak{T})$, $(l_i : i \in \omega)$ are the **triple of partition, coefficients and parameters** *for the group X.*

Let $K = \bigoplus_{i\in\omega} K^{l_i}(A_i)$ be a subgroup of X with $K(A_i) = \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau \rangle \subset A_i = \bigoplus_{\tau\in T_i} \tau a_\tau$.

Then $B = X/K$ is a **proper $\mathfrak{B}^{(1)}$alr-group**.

Direct decomposition theory of almost rigid groups was investigated in [7]. It shows that among different direct decompositions there exist the so-called **main decompositions**:

Definition 5.7. A special decomposition $X = Y \oplus C$ with rigid Y and completely decomposable C satisfying the conditions

(1.) $\tau \in T_{cr}(Y)$ if and only if $m_\tau(X) > 1$;

(2.) $m_\tau(Y) = m_\tau(X)$ for any $\tau \in T_{cr}(Y)$

will be called **a main decomposition** of the almost rigid group X.

Let X be an almost rigid group from the class $\mathcal{H}'_{\mathfrak{T}}$ and

$$X = \bigoplus_{j \in I} X_j \oplus X' \tag{14}$$

be a direct decomposition of X, where X' is a completely decomposable group and all X_j are almost rigid groups (it is proved in [7] that the class of alr-groups is closed under taking direct summands if we naturally accept that a block-rigid completely decomposable group belongs to this class).

Definition 5.8. Let an almost rigid group X from the class $\mathcal{H}'_{\mathfrak{T}}$ have a decomposition (14). If X_j $(j \in I)$ are rigid groups, having regulators isomorphic to direct sums of groups from the set $\{A_i,\ i \in \omega\}$ and for each $T_i \subset T_{cr(X_j)}$ there exists $\tau \in T_i$ such that $m_\tau(X_j) \neq 1$ then the decomposition (14) will be called an **admissible decomposition** of X.

Definition 5.9. An admissible decomposition $X = Y_1 \oplus C_1$, where Y_1 is a rigid group satisfying:

1. $T_{cr}(Y_1) \supseteq T_i$ if and only if $m_\tau(X) \neq 1$ for some $\tau \in T_i$;
2. $m_\tau(Y_1) = m_\tau(X)$ for all $\tau \in T_{cr}(Y_1)$

will be called a **main admissible decomposition** of X.

Clearly, the direct summand C_1 of a main admissible decomposition is completely decomposable.

We will now develop the direct decomposition theory for $\mathfrak{B}^{(1)}$alr-groups based on the admissible decompositions for almost rigid groups. We also need to apply the following number theory result, which derives from the classical Chinese Remainder Theorem and has been already used (in a slightly different form) for studying endomorphism rings of rigid crq-groups, see [6, Lemma 6.3].

Lemma 5.10. *For any finitely many positive numbers $s_i \in \mathbb{Z}$, $p_i \in \mathbb{Z}$ $(i \in I)$ and $n \in \mathbb{Z}$ with $\gcd(p_i, p_j) = 1$ if $i \neq j$ and $\gcd(s_i, n) = 1$ there exists a positive number $x \in \mathbb{Z}$, for which $x \equiv s_i \pmod{p_i}$ and $\gcd(x, n) = 1$.*

Proof. We induct on I. The induction starts with the case $|I| = 2$. Find a number x satisfying $x \equiv s_1 \pmod{p_1}$ and $x \equiv s_2 \pmod{p_2}$. Define $p = p_1$ and $q = p_2$, then $\gcd(p, q) = 1$. Hence x is expressed as

$$x = s_1 + kp \qquad \text{and} \qquad x = s_2 + lq \tag{15}$$

for some k, l. There are integers u, v such that $pu + qv = 1$, write $(s_2 - s_1)pu + (s_2 - s_1)qv = s_2 - s_1$. Then $s_2 - u(s_2 - s_1)p = s_1 + v(s_2 - s_1)q$. Denote $k = u(s_1 - s_2)$, $l = v(s_2 - s_1)$ and take

$$\overline{x} = s_2 + kp = s_1 + lq. \tag{16}$$

(In the case $s_1 = s_2$ we simply put $\overline{x} = s_1 = s_2$).

If $\gcd(\overline{x}, n) = 1$, set $x = \overline{x}$. Otherwise denote $r_i \quad (i \leqslant s)$ the prime divisors of n such that $\overline{x}$ is divisible by r_i. The other prime divisors of n will be denoted by $f_j \quad (j \leqslant t)$. Since $\gcd(s_i, n) = 1 \quad$ (i=1, 2), $\quad kp$ and lq are relatively prime to all r_i by (16), then

$$\gcd(pq, r_i) = 1. \tag{17}$$

Clearly

$$x = \overline{x} + fpq \tag{18}$$

satisfies (15) for any integer f. Define $f = \prod_{j \leqslant t} f_j$, then $\gcd(x, n) = 1$, because $\gcd(r_i, f_j) = 1$, $i \leqslant s$, $j \leqslant t$ implies that for any prime divisor r of n exactly one summand in (18) is divisible by r. (If x turned out to be a negative integer, we can take a suitable power of f instead of f in (18)). This proves the claim in the primary case.

By induction hypothesis there is a number s such that $s \equiv s_i \pmod{p_i}$, $i < m$, and

$$\gcd(s, n) = 1 \tag{19}$$

Denote $p = \prod_{i<m} p_i$ and $q = p_m$. Clearly the required x satisfies $x = s + kp$ and $x = s_m + lq$ for some k and l. If we denote $s_1 = s$ and $s_2 = s_m$ at this stage, we exactly obtain (15). Repeating the proof as in the primary case we complete the proof of the Lemma. □

Remark 5.11. If n is relatively prime to every $p_i \quad (i \in I)$, then the condition $\gcd(s_i, n) = 1$ is not necessary because (17) is satisfied automatically.

Another simple number theory result will be used in the proofs referring to $\tau \in T_{cr}(X)$ for almost rigid group X.

Lemma 5.12. *For any reduced rational fraction $\frac{s}{k} \in \tau$ and natural number m satisfying $\tau(p) \neq \infty$ for each prime divisor p of m there exists a natural number c such that $c - \frac{s}{k} \in m\tau$.*

Proof. (Routine.) Let $c_0 = \gcd(s, m)$ and c' be a member of the congruence class $\overline{s'}(\overline{k})^{-1} \pmod{\frac{m}{c_0}}$ with $s = c_0 s'$ (recall that $\gcd(k, m) = 1$). Set $c = c_0 c'$, then $c - \frac{s}{k} = \frac{ck-s}{k} = \frac{c_0(c'k-s')}{k} \in m\tau$.

□

Theorem 5.13. *Let $\phi : X \longmapsto B$ be a regular representation of a $\mathfrak{B}^{(1)}$alr-group B with the triple $(T_i : i \in \lambda)$, $(\alpha_\tau : \tau \in \mathfrak{T})$ and $(l_i : i \in \omega)$ of partition, coefficients and parameters for the group X. Then there exists a main admissible decomposition $X = X' \oplus F$ with $m_\tau(X') = m_\tau(X)$ and the corresponding decomposition $B = B' \oplus H$ such that $\phi(X') = B'$, $\phi(F) = H$.*

Proof. First, we concentrate on the group X. Recall that for the partition $T_{cr}(X) = T = \bigcup_{i\in\omega} T_i$ and invariants $m_\tau = m_\tau(X)$ the following holds: $\gcd(m_\tau, m_\sigma) = 1$ if $\tau, \sigma \in T_i$ for some i, $\tau \neq \sigma$. Furthermore $R(X) \cong \bigoplus_{i\in\omega} A_i^{l_i}$ where A_i are rigid groups of ranks k_i with critical typesets $T_{cr}(A_i) = T_i$.

Let $\{a_\tau^n : \tau \in T_i,\ n = 1, \dots, l_i,\ i \in \omega\}$ be a K-basis of X. Direct summands of $\operatorname{Ker}\phi = K \cong \bigoplus_{i\in\omega} K^{l_i}(A_i)$ can be expressed as $K^{l_i}(A_i) = \bigoplus_{n=1}^{l_i} \langle \sum_{\tau\in T_i} \alpha_\tau a_\tau^n \rangle$ with

$$X(\tau) = \bigoplus_{n\leqslant l_i} \tau a_\tau^n, \quad \tau \in T_i. \tag{20}$$

Let $X = X_0 \oplus Y_0$ be any main admissible decomposition of X, where X_0 is a rigid group with $m_\tau = m_\tau(X_0) = m_\tau(X)$ ($\tau \in T_{cr}(X_0)$). Assume that $T_i \subset$

$T_{cr}(X_0)$ if and only if $i \in \mathcal{I}$ for some $\mathcal{I} \subset \omega$ and denote $T' = T_{cr}(X_0) = \bigcup_{i\in\mathcal{I}} T_i$ (in other words there exists $m_\tau(X) \neq 1$ for some $\tau \in T_i$ if and only if $i \in \mathcal{I}$). Then

$$R(X_0) = \bigoplus_{i\in\mathcal{I}} (\bigoplus_{\tau\in T_i} \tau a_\tau) \tag{21}$$

for some $a_\tau \in X(\tau)$. Since $X_0(\tau) = \tau a_\tau \subset X(\tau)$, from (20) we obtain that

$$a_\tau = \sum_{n=1}^{l_i} d_{\tau n} a_\tau^n, \quad \tau \in T_i \subset T_{cr}(X_0) \tag{22}$$

where $d_{\tau n} = \frac{s_{\tau n}}{k_{\tau n}}$ are reduced fractions such that $\tau(p) = \infty$ for every prime divisor p of $n_{\tau n}$.

Let

$$a'_\tau = \sum_{n=1}^{l_i} c_{\tau n} a_\tau^n, \quad \tau \in T_i \tag{23}$$

with integer coefficients $c_{\tau n}$ satisfying $c_{\tau n} - \frac{s_{\tau n}}{k_{\tau n}} \in m_\tau \tau$, for $\tau \in T_i, \quad i \in \mathcal{I}$, see Lemma 5.12, Remark 5.5.

Next we will apply Lemma 5.10 l_i times for each i. Our aim is to find for every $i \in \mathcal{I}$ the integers c_n^i such that $c_n^i \equiv c_{\tau n} \pmod{m_\tau}$, $\tau \in T_i$ and $\gcd((c_1^i, \ldots, c_{l_i}^i)) = 1$ (it is not necessary if for some n the numbers $c_{\tau n}$, $\tau \in T_i$, in (23) are equal to the same number, say c_n^i, or $l_i = 1$). The numbers $c_n^i \equiv c_{\tau n} \pmod{m_\tau}$ exist, because the conditions of Lemma 5.10 are satisfied by Definition 5.6(6.) It remains to prove that under a special choice of them the condition on the greatest common divisor also holds.

Fix any $i \in \mathcal{I}$ and denote $c_n = c_n^i$ to simplify our notation. Let $d = \gcd(c_1 \ldots, c_{l_i})$ and $m = \operatorname{lcm}_{\tau\in T_i} m_\tau$. Since X_0 is a direct summand of X, $X_0(\tau)$ is pure in X and the numerators $s_{\tau n}$ of the coefficients $d_{\tau n}, \quad n = 1, \ldots, l_i,$ in (22) have a prime common divisor p only if $\tau(p) = \infty$, and $\gcd(p, m_\tau) = 1$ for each $\tau \in T_i$ by Definition 5.6(8.). Then

$$\gcd(c_{\tau 1}, \ldots, c_{\tau l_i}),\ \tau \in T_i, \quad \text{is relatively prime with} \quad m_\tau.$$

We claim that $\gcd(d, m) = 1$. Otherwise there exists a prime q and $\sigma \in T_i$ such that $q|m_\sigma$ and q divides $c_1, \ldots, c_{l_i}$. Since $c_n \equiv c_{\sigma n} \pmod{m_\sigma}$ and $q|c_n$ we obtain that all $c_{\sigma n}$ $(n = 1, \ldots, l_i)$ are q-divisible. As $q|m_\sigma$, it yields a contradiction, which leads us to the conclusion $\gcd(d, m) = 1$. !In fact,! we can choose the numbers !satisfying! $\gcd(c_1, \dots, c_{l_i}) = 1$. Take the maximal $n \in \mathbb{N}$ dividing each c_n, $n = 2, \dots, l_i$ and satisfying $\gcd(n, m) = 1$. Then we apply Lemma 5.10 to the numbers $c_{\tau 1}$, $\tau \in T_i$. Since n is relatively prime to each m_τ, we can find a new positive integer $c_1 \equiv c_{\tau 1} \pmod{m_\tau}$, $\tau \in T_i$, such that $\gcd(c_1, n) = 1$ by Remark 5.11 and Definition 5.6(6.). Thus we may assume that the integers c_n, $n = 1, \dots, l_i$ have no nontrivial common divisors.

Return to the c_n^i. We now may assume that $c_n^i \equiv c_{\tau n} \pmod{m_\tau}$, and $\gcd((c_1^i, \dots, c_{l_i}^i)) = 1$, $i = 1, \dots, k$. Consider the elements

$$\widetilde{a_\tau} = \sum_{n=1}^{l_i} c_n^i a_\tau^n \tag{24}$$

with $\tau \in T_i$ and consider rigid completely decomposable groups

$$V_i = \bigoplus_{\tau \in T_i} \tau \widetilde{a_\tau}, \quad i \in \mathcal{I}, \tag{25}$$

which are pure in X by condition $\gcd((c_1^i, \dots, c_{l_i}^i)) = 1$, and Definition 5.6(6.). Note that the coefficients c_n^i in (24) are the same for all $\tau \in T_i$. Define a rigid subgroup of X:

$$X' = \left(\bigoplus_{i \in \mathcal{I}} V_i \right)_*^X = \left(\bigoplus_{\tau \in T'} \tau \widetilde{a_\tau} \right)_*^X, \tag{26}$$

by construction, $\widetilde{a_\tau} \equiv a_\tau \pmod{m_\tau V_\tau}$. Then $m_\tau = m_\tau(X') = m_\tau(X_0) = m_\tau(X)$ for all $\tau \in T' = T_{cr}(X_0) = T_{cr}(X')$.

For every $i \in \mathcal{I}$ construct a matrix D_i from the ring $\mathrm{Mat}_{(l_i \times l_i)}(\mathbb{Z})$ with $\det D_i = 1$, whose first column coincides with $(c_1^i, \dots, c_{l_i}^i)^T$, see Lemma 4.3. Set $D_i = E$ (the unit-matrix) for $i \in \mathfrak{T} \setminus \mathcal{I}$, that corresponds to T_i with $m_\tau(X) = 1$ for every $\tau \in T_i$. Thus we have a K-basic automorphism Ψ of X, which acts on $X(\tau)$, $\tau \in T_i$, as multiplication by the same matrix D_i, $i \in \omega$, with respect to $\{a_\tau^n : n = 1, \dots, l_i\}$ and leads to the main admissible decomposition $X = X' \oplus F$ with the second summand a completely decomposable group. Then $K = K_1 \oplus K_2$ where $K_1 \subset X'$, $K_2 \subset F$ and $B = X/K = X'/K_1 \oplus F/K_2$ is a desired decomposition of B. We complete the proof setting $B' = X'/K_1 = \phi(X')$ and $H = F/K_2 = \phi(F)$. □

Definition 5.14. A decomposition $B = B' \oplus H$ with a rigid $\mathfrak{B}^{(1)}$alr-group B' such that $B/R(B) \cong B'/R(B')$ will be called a **main decomposition of the $\mathfrak{B}^{(1)}$alr-group** B.

We see that $\mathcal{A}$ is a strongly decomposable block-rigid group as it is a direct sum of proper $\mathfrak{B}^{(1)}$-groups, which are strongly indecomposable, and $\mathfrak{T} = T_{cr}(\mathcal{A})$ is an antichain (we accept here that some l_i can be equal to 1 and $K(A_i) = 0$ in the particular case, only $l_i = 2$ is excluded). According to Definition 5.1, under the condition of purity V_i in X, $i \in \mathcal{I}$, (see (25)), we have that $\phi : X' \longmapsto B'$ is a regular representation of the rigid $\mathfrak{B}^{(1)}$alr-group B' which is also called a rigid proper local $\mathfrak{B}^{(1)}$asd-group. Since $X' \in \mathcal{H}_{\mathfrak{T}}$ we may apply Definition 3.1 and Lemma 3.6 to conclude that groups $\phi(V_i) \cong A_i/K(A_i)$ are pure in B and B', which is the reason to denote $R(B') = \phi(\bigoplus_{i\in\mathcal{I}} V_i)$ and call this group the **regulator** of B'. We have $R(B')(\tau)$ is pure in B for each $\tau \in T'$.

Furthermore, $\phi(F) \cong (\bigoplus_{i\in\mathcal{I}}(A_i/K(A_i))^{l_i-1}) \oplus (\bigoplus_{i\in\omega\setminus\mathcal{I}}(A_i/K(A_i))^{l_i})$ is a strongly decomposable group, hence $\mathcal{A} = R(B') \oplus \phi(F) \cong \bigoplus_{i\in\omega}(A_i/K(A_i))^{l_i}$ satisfies the property

$$\mathcal{A}(\tau) \quad \text{is a pure subgroup of} \quad B. \tag{27}$$

Again, following !the! traditional terminology deriving from acd-group theory, we call $\mathcal{A}$ the **regulator** of B and denote it by $R(B)$.

Furthermore, $X/R(X) \cong (X/K)/(R(X)/K) = B/\mathcal{A}$ is a direct sum of p-primary cyclic groups ($p \in P_X$). The name "proper $\mathfrak{B}^{(1)}$alr-group" shows that such a group B differs from an almost rigid group by construction of the regulator, which is a direct sum of countably many $\mathfrak{B}^{(1)}$-groups in comparison with a completely decomposable group as !the! regulator of an almost rigid group (note, that in the both cases the regulators are strongly decomposable).

We are now ready to prove a classification theorem of $\mathfrak{B}^{(1)}$alr-group.

Theorem 5.15. *Let B and C be proper $\mathfrak{B}^{(1)}$alr-groups and let $\phi : X \longmapsto B$ and $\psi : Y \longmapsto C$ be their regular representations respectively. Then $B \cong_{nr} C$ if and only if $X \cong_{nr} Y$ and the triples of partition, coefficients and parameters for X and Y coincide.*

Proof. Let $B \cong_{nr} C$ and apply Definition 2.3. For a fixed $p \in P_X$ the monomorphisms $\eta_p \;: B \longrightarrow C$ and $\mu_p \;: C \longrightarrow B$ with zero p-components of torsion groups $C/B\eta_p$ and $B/C\mu_p$, satisfy the property for $R(C)_\tau/R(B)_\tau\eta_p$ and $R(B)_\tau/R(C)_\tau\mu_p$ to be finite subgroups of $C/B\eta_p$ and $B/C\mu_p$ accordingly for any $\tau \in T_{cr}(B)$ and $\tau \in T_{cr}(C)$ by (27). Therefore, $T_{cr}(B) = T_{cr}(C)$ and also $T_{cr}(X) = T_{cr}(Y)$. !Moreover, we have $R(C)(T_i) \cong_{nr} R(B)(T_i)$ as finite rank groups which implies $R(C)(T_i) \cong R(B)(T_i)$ for all $i \in \omega$, because a finite index subgroup of $A_i/K(A_i))^{l_i}$ is isomorphic to the group itself.!

Then $R(C) \cong R(B)$. This yields $R(X) \cong R(Y)$ and the regular representations ϕ and ψ have the same triples of partition, coefficients and parameters by Remark 4.2.

Furthermore, take the restriction of η_p to $(\bigoplus_{\tau\in[T_p^X]} B(\tau))_*{}^B$, the image of this group is a finite index subgroup of $(\bigoplus_{\tau\in[T_p^Y]} C(\tau))_*{}^C$ by Definition 2.3 (3.) and Definition 2.1, because the sets $[T_p^Y]$, $[T_p^X]$ are finite for almost rigid groups. This means that $B'_p \cong_{nr} C'_p$ as $\gcd(|C'_p/B'_p\eta_p|, p) = 1$ (recall that for finite rank groups the near-isomorphism is ensured by the one-sided monomorphisms, see Introduction). We have $X'_p \cong_{nr} Y'_p$ because $B'_p \cong X'_p$ and $C'_p \cong Y'_p$. Then $X_p \cong_{nr} Y_p$ for all prime p and by Remark 3.4 we conclude that $X \cong_{nr} Y$ as required.

Conversely, let $X \cong_{nr} Y$. Then $T_{cr}(X) = T_{cr}(Y) = \bigcup_{i\in\omega} T_i$. Based on Theorem 5.13, from the existence of the main admissible decompositions $X = X' \oplus F$ and $Y = Y' \oplus D$ of the groups $X' \cong_{nr} Y'$ such that $B = B' \oplus F\phi$ and $C = C' \oplus F\phi$ with rigid proper $\mathfrak{B}^{(1)}$alr-groups $B' = \phi(X')$ and $C' = \psi(Y')$ we have $B' \cong_{nr} C'$ (see Theorem 3.10). !Moreover, $F\phi$ and $D\psi$ are isomorphic as strongly decomposable groups by Remark 4.2.! Then $B \cong_{nr} C$.

□

Remark 5.16. It can be seen from the proof of Theorem 5.15 that a similar classification can be obtained on the basis of Theorem 3.10 for a wider class of almost strongly decomposable groups B if they have a main decomposition $B = B' \oplus H$ with a rigid first summand such that $B/R(B) \cong B'/R(B')$.

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