Breaking up finite automata presentable torsion-free abelian groups*

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In [13] it was shown that the group of rational numbers is not FA-presentable, i.e. it does not admit a presentation by a finite automaton. More generally, any torsion-free abelian group that is divisible by infinitely many primes is not of this kind. In this article we extend the result from [13] and prove that any torsion-free FA-presentable abelian group G is an extension of a finite rank free group by a finite direct sum of Prüfer groups $\mathbb{Z}(p^\infty)$.

Keywords: FA-presentable abelian groups, automatic structures, additive combinatorics

1. Introduction

Let Σ be a finite alphabet and let Σ^* denote the full language formed by Σ, i.e. the set of all finite words built from Σ. Recall that a language (a subset of Σ^*) is *regular* if there exists a finite automaton recognizing it.

A relational structure $(M; R_1, \ldots, R_k)$ is *FA-presentable* (also called automatic or regular but these are also used in different meanings) if there exists a bijection $g\colon D \cong M$ with a regular language D such that the relations $g^{-1}(R_1), \ldots, g^{-1}(R_k)$ are also regular. Here the $g^{-1}(R_i)$ are considered as a subset of $(\Sigma \cup \{\diamond\})^*$ encoded via padding and convolution: Every element of $g^{-1}(R_i)$ is a tuple of words. By padding, adding sufficiently many $\diamond$ to the shorter words, we make all words have the same length. Finally, we interleave the words to obtain a single word, their convolution. (For a precise definition, see any of [10, Definition 1], [5, Definition

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1.4, automatic presentation] or [7].)

Functions are represented in regular relational structures via their graphs. This is significant to us, as we are interested in algebraic structures.

For the basic theory of FA-presentable structures, see [5]. We just mention that the first order theory and model checking of FA-presentable structures are decidable (see [11], [4]).

FA-presentability is a rather restrictive property. For rich algebraic structures, the FA-presentable ones are often only the obvious ones. For example, the FA-presentable Boolean algebras are exactly the finite ones and finite powers of the algebra of finite and co-finite subsets of $\mathbb{N}$. Similarly, the FA-presentable integral domains (with the ring structure) are exactly the finite ones ([7], [10]).

FA-presentability is restrictive even for simple algebraic structures like groups: these groups are all locally virtually abelian, i.e. every finitely generated subgroup has an abelian subgroup of finite index [10, Theorem 10].

As a consequence, it is natural to consider FA-presentability only among abelian groups. On the one hand, there are already some interesting known examples: finite groups, $\mathbb{Z}$, the Prüfer group $\mathbb{Z}(p^\infty)$, $\mathbb{Z}[1/n]$, countably infinite direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$. Most of these are based on the usual school algorithm of addition with carry.

Furthermore, the class of FA-presentable groups is closed under finite sums [3, Theorem 2] (so all finitely generated abelian groups are FA-presentable), finite extensions and, more interestingly, FA-presentable amalgamation [9].

On the other hand, there are few non-trivial examples of abelian groups known not to be FA-presentable, like the ones containing a free abelian group of infinite rank [7] (see also [10, Theorem 5]), the group of rational numbers and more generally any torsion-free abelian group that is divisible by infinitely many primes, and torsion groups of the form $\bigoplus_{p\in I} \mathbb{Z}(p^\infty)$, where I is an infinite set of primes (a recent result of Tsankov [13, Theorem 2]).

In this paper, we improve the results and show in Theorem 10 that every FA-presentable torsion-free abelian group is an extension of a finite-rank free group by a finite direct sum of Prüfer groups. In particular, the rings $\mathbb{Z}[1/n]$ are the only FA-presentable torsion-free abelian groups of rank 1. The idea of the proof is to show a finite and local version of the main result from [13]. Basic knowledge of abelian groups is assumed, approximately [2, Chapter I, III, Sections 8, 9, 25, 26, 27].

See the survey article [6] for open questions of FA-presentable (called string automatic) and other automatic structures. In particular, Questions 2.3, 4.4 and 4.7 there ask characterization of FA-presentable torsion-free abelian groups and FA-presentable isomorphisms between them.

2. Outline

In Section 3 we recall the additive combinatorics used for proving that $\mathbb{Q}$ is not FA-presentable in [13]. We use this to obtain a local version for finite groups in Section 4. After these preparations, we will be ready to analyse FA-presentable torsion-free abelian groups in Section 5 breaking them up as extensions in the end.

Throughout the paper, we will use the following standard notation. For a finite set A, let $|A|$ denote its size. Let $\langle A \rangle$ be the subgroup generated by A. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of the natural numbers, the integers, the rationals, and the reals, respectively.

A key ingredient in [13] and hence our investigation, which also motivates some of the technical results, is

Lemma 1. *[13, top half of page 5] For every FA-presentable abelian group G, there exist a sequence of finite subsets A_n of G, a constant C_1 and a function $h\colon \mathbb{N} \to \mathbb{N}$ such that:*

(i) $\bigcup_{n=0}^{\infty} A_n = G$;
(ii) $0 \in A_0$ *and* $|A_0| \geq 2$;
(iii) $A_n + A_n \subseteq A_{n+1}$;
(iv) $|A_{n+1}| \leq C_1 |A_n|$;
(v) For every $x \in A_n$ *and* $m \in \mathbb{N}$ *with* $m \mid x$ *in* G *there is a* $y \in A_{n+h(m)}$ *with* $x = my$. *(If G is torsion-free, this means* $m^{-1}A_n \subseteq A_{n+h(m)}$.*)*

(Condition (i) is implicit in the reference.) Obviously, in (v) it is enough to consider only prime numbers m, since it implies the condition for every number.

3. Additive combinatorics

In this section we recall some parts of [13] with minor modifications and a bit generalization. Our aim is to prove the crucial estimation Lemma 3 for our finite version Proposition 4 of $\mathbb{Q}$ not being FA-presentable in the next section.

The main reference for additive combinatorics is the book by Tao and Vu [12].

Let G be an abelian group. A *generalized arithmetic progression* (or just a progression, for short) in G is a pair (P, ϕ), where P is a finite subset of G and ϕ is an affine map from a parallelepiped in $\mathbb{Z}^d$ onto P, i.e.,

$$P = \left\{ v_0 + \sum_{i=1}^{d} a_i v_i : 0 \leq a_i < N_i \text{ for } i = 1, \dots, d \right\},$$

where $v_0, v_1, \dots, v_d \in G$, and $N_1, \dots, N_d \in \mathbb{N}$, with $\phi(a_1, \dots, a_d) = v_0 + \sum_{i=1}^{d} a_i v_i$. The number d is called the *rank* of the progression, e.g. progressions of rank 1 are the arithmetic progressions. A progression is called *proper* if ϕ is injective. We recall Freiman's theorem.

Theorem 2 (Freiman's theorem [1]). *Let $C_1 > 0$ be a constant. Then there exist constants K and d such that for every finite set $A \subseteq G$ of a torsion-free* *abelian group G satisfying $|A+A| \leq C_1 |A|$, there exists a proper progression P of rank at most d that contains A and $|P| / |A| \leq K$.*

To measure how close a finite set A is to a progression of rank d, define

$$\theta(A,d) := \min \left\{ \frac{|P|}{|A|} : P \supseteq A \text{ and } P \text{ is a proper progression of rank } \leq d \right\}.$$

If there is no d-dimensional progression covering A, put $\theta(A,d) := \infty$. Clearly, θ satisfies monotonicity:

$$B \subseteq A \implies \theta(A,d) \geq \frac{|B|}{|A|} \theta(B,d). \tag{3}$$

Intuitively, adding an element not divisible by p to a progression all of whose elements are divisible by p, the rank will increase. This is precisely formulated as follows.

Lemma 3 (C.f. [13, Lemma 8]). *Let $d \geq 1$ be an integer and $p > d!$ be a prime. Let G be an abelian group. Let $A \subseteq G$ be a finite set with at least 2 elements, and $z \in G$ have its order in $G/\langle A\rangle$ divisible by p. Then*

$$\theta(A \cup \{z\}, d) \geq \min \left\{ \frac{p^{1/d}}{4d}, \frac{\theta(A, d-1)}{d^{C_0 d^3}} \right\}, \tag{4}$$

where C_0 is an absolute constant.

Proof. The proof is essentially the same as the proof of [13, Lemma 8] with the only significant modification that we replace "$\|X\|_p < \|z\|_p$" by "the order of z in $G/\langle X\rangle$ is divisible by p". (This is what we have done in the statement of the lemma.) We exemplify this on the beginning of the proof.

Let P be a proper progression

$$P = \left\{ v_0 + \sum_{i=1}^{d} a_i v_i : 0 \leq a_i < N_i \text{ for } i = 1, \ldots, d \right\},$$

of rank d covering $A \cup \{z\}$. In the group $G/\langle A\rangle$, the element z is contained in P, which is contained in a coset of $\langle v_1, \ldots, v_d\rangle$. This coset is actually the subgroup $\langle v_1, \ldots, v_d\rangle$ itself, as it contains 0 (the image of A). Containing an element whose order is divisible by p, there must be a generator with order also divisible by p.

By reordering $v_1, \ldots, v_d$, we can assume that there exists $k \geq 1$ such that

$$v_1, \ldots, v_k \notin \langle A\rangle, \quad v_{k+1}, \ldots, v_d \in \langle A\rangle$$

and the order of v_1 in $G/\langle A\rangle$ is divisible by p.

Let the lattice Γ in $\mathbb{R}^k$ given by

$$\Gamma = f^{-1}(\langle A\rangle) = \left\{ x \in \mathbb{Z}^k : f(x) \in \langle A\rangle \right\}.$$

One now proceeds as in [13, Lemma 8]. □

4. Finitary versions

Now we recall the essence of the proof of $\mathbb{Q}$ not being FA-presentable. We state a *finite* version of this theorem that will then be used to prove a *local* result replacing the group of rational numbers by the finite group $\mathbb{Z}/p\mathbb{Z}$ for any prime p.

Proposition 4 (C.f. [13, Theorem 7]). *Given a constant C_1 there are integers $d, K \in \mathbb{Z}^+$ and a constant $C \geq C_1$ such that the following hold for any sequence $p_0, \dots, p_d$ of primes and integers $h(p_i)$ with*

$$p_{i-1} > p_i C^{h(p_i)d} d^{Cd^4} \qquad i = 1, \dots, d \tag{8}$$

$$p_d > C(4dK)^d. \tag{9}$$

There is no sequence $A_0, \dots, A_{h(p_0)+\dots+h(p_d)+1}$ of finite subsets of a torsion-free abelian group G such that

- *(Q-i)* $0 \in A_0$ *and* $|A_0| \geq 2$;
- *(Q-ii)* $|A_{n+1}| \leq C_1 |A_n|$ *for* $n \leq h(p_0) + \dots + h(p_d)$;
- *(Q-iii)* $A_n + A_n \subseteq A_{n+1}$ *for* $n \leq h(p_0) + \dots + h(p_d)$;
- *(Q-iv)* $\langle A_n \rangle \cap p_i G \subseteq p_i \langle A_{n+h(p_i)} \rangle$ *and* $p_i \langle A_n \rangle \subsetneq \langle A_n \rangle \cap p_i G$ *for* $n + h(p_i) \leq h(p_0) + \dots + h(p_d)$.

Proof. The proof is essentially the same as that of [13, Theorem 7]. Let $C := \max\{C_0, C_1\}$, where C_0 is the constant from Lemma 3. By Freiman's theorem, there exist constants K and d such that $\theta(A, d) \leq K$ for all set A with $|A + A| \leq C_1 |A|$.

Suppose for contradiction that there is a sequence of primes $p_d < p_{d-1} < \dots < p_0$ satisfying the conditions

$$\begin{aligned} p_d &> C(4dK)^d, \\ p_{i-1} &> p_i C^{h(p_i)d} d^{Cd^4} \quad i = d, d-1, \dots, 1 \end{aligned} \tag{10}$$

for a sequence A_n satisfying (Q-i), (Q-ii), (Q-iii) and (Q-iv).

Define inductively the sequence of integers $n_0 < n_1 < \dots < n_d$ by

$$n_0 = 0 \quad \text{and} \quad n_i = \min \left\{ n : t_{p_i} \frac{\langle A_n \rangle}{\langle A_{n_{i-1}} \rangle} \neq 0 \right\} \quad \text{for } i = 1, \dots, d.$$

Note that the quotient $\langle A_{n_{i-1}+h(p_i)} \rangle / \langle A_{n_{i-1}} \rangle$ has non-trivial p_i-torsion part, as it is isomorphic to $p_i \langle A_{n_{i-1}+h(p_i)} \rangle / p_i \langle A_{n_{i-1}} \rangle$, which contains the non-trivial p_i-group $\langle A_{n_{i-1}} \rangle \cap p_i G / p_i \langle A_{n_{i-1}} \rangle$ by property (Q-iv) of the family $\{A_n\}$. Thus $n_i \leq n_{i-1} + h(p_i)$ and hence, first, n_d is at most $\sum_{i=0}^d h(p_i)$, and second,

$$|A_{n_i - 1}| \leq C^{h(p_i)-1} \left| A_{n_{i-1}} \right|. \tag{12}$$

We will prove by induction on i that

$$\theta(A_{n_i}, i) > C^{-1} p_i^{1/d} / (4d) \quad \text{for all } i = 0, \dots, d. \tag{13}$$

Applied for $i = d$, this will yield a contradiction with the choice of p_d. The case $i = 0$ follows trivially from the definition of θ. Suppose now that $i \geq 1$ and (13) holds for $i-1$ in order to prove it for i. By the induction hypothesis, (12), and (3),

$$\theta(A_{n_i-1}, i-1) \geq C^{-h(p_i)+1}\theta(A_{n_{i-1}}, i-1) > C^{-h(p_i)} p_{i-1}^{1/d}/(4d). \tag{14}$$

By the choice of n_i, there exists $z \in A_{n_i}$ having order divisible by p_i in $G/\langle A_{n_{i-1}} \rangle$. Apply Lemma 3 to the set $A_{n_i-1} \cup \{z\}$ and the prime p_i to obtain

$$\begin{aligned} \theta(A_{n_i}, i) &> C^{-1}\theta(A_{n_i-1} \cup \{z\}, i) \\ &\geq C^{-1} \min\left\{p_i^{1/d}/(4d), \theta(A_{n_i-1}, i-1)/d^{Cd^3}\right\}. \end{aligned}$$

The choice of p_{i-1} and (14) allow us to conclude that $p_i^{1/d}/(4d) \leq \theta(A_{n_i-1}, i-1)/d^{Cd^3}$, which completes the induction and the proof. □

We now replace the torsion-free G by the finite group $\mathbb{Z}/p\mathbb{Z}$ for a large prime p.

Proposition 5. *Given a constant C_1 there are integers $d, K \in \mathbb{Z}^+$ and a constant $C \geq C_1$ such that the following hold for any sequence $p_0, \dots, p_d$ of primes and integers $h(p_i)$ with*

$$p_{i-1} > p_i C^{h(p_i)d} d^{Cd^4} \qquad i = 1, \dots, d \tag{16}$$

$$p_d > C(4dK)^d. \tag{17}$$

There is a p^* (depending also on the p_i and $h(p_i)$) such that for any prime $p \geq p^*$, there is no sequence $A_0, \dots, A_{h(p_0)+\dots+h(p_d)+1}$ of finite subsets of $\mathbb{Z}/p\mathbb{Z}$ such that

- *(F-i)* $0 \in A_0$, $2 \leq |A_0| \leq C_1$
- *(F-ii)* $|A_{n+1}| \leq C_1 |A_n|$ *for* $n \leq h(p_0) + \dots + h(p_d)$
- *(F-iii)* $A_n + A_n \subseteq A_{n+1}$ *for* $n \leq h(p_0) + \dots + h(p_d)$
- *(F-iv)* $p_i^{-1} A_n \subseteq A_{n+h(p_i)}$ *for* $n + h(p_i) \leq h(p_0) + \dots + h(p_d)$ *and* $i = 0, \dots, d$.

We thank for the referee for suggesting a significant shortening of our original proof.

Proof. We show that for C_1, we can use the same d, K and C as in Proposition 4. Let $p_0, \dots, p_d$ be primes and $h(p_i)$ be integers satisfying (16) and (17).

It is easy to obtain a constant upper bound C_2 of the size of $\bigcup_i A_i$ depending only on C_1 and $h(p_0), \dots, h(p_d)$ via (F-i) and (F-ii). Let $k := \max\{2, p_0, \dots, p_d\}$. By [12, Exercise 5.3.11], for $p \geq \log_{2k} C_2 =: p^*$, the set $\bigcup_i A_i$ is Freiman isomorphic to a subset A' of $\mathbb{Z}$ of order $\max\{2, p_0, \dots, p_d\}$. (The isomorphism is actually the canonical projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.) Recall that a Freiman isomorphism of order k is an isomorphism φ satisfying

$$a_1 + \dots + a_k = b_1 + \dots + b_k \iff \varphi(a_1) + \dots + \varphi(a_k) = \varphi(b_1) + \dots + \varphi(b_k)$$

for every $a_1, \ldots, a_k, b_1, \ldots, b_k$ in the domain. By translating A' if necessary, we may assume that the image of $0 \in \bigcup_i A_i$ is 0. Let A'_n be the image of A_n.

Via the Freiman isomorphism, obviously the conditions (Q-i) to (Q-iv) of Proposition 4 hold for the finite sequence $\langle A'_n : n \le H \rangle$ considered as subsets of $\mathbb{Q}$. This contradicts Proposition 4 and hence finishes the proof. □

5. Properties of FA-presentable groups

Building on the previous technical results, in this section we gradually establish algebraic properties of FA-presentable abelian groups culminating in our main theorem.

We start by an obvious application of Freiman's theorem.

Lemma 6. *Every FA-presentable torsion-free abelian group has finite rank.*

Namely, every finite subset is contained in a proper progression of rank at most d and hence in a subgroup of rank at most d. Therefore the rank of the group is at most d.

We now strengthen Lemma 1 by formulating the divisibility condition (ii) for subgroups in (G-v).

Lemma 7. *For every FA-presentable torsion-free group G, there exist a sequence of finite subsets A_n of G, a constant C_1 and a function $h\colon \mathbb{N} \to \mathbb{N}$ such that:*

(G-i) $\bigcup_{n=0}^{\infty} A_n = G$;
(G-ii) $0 \in A_0$ *and* $|A_0| \ge 2$;
(G-iii) $A_n + A_n \subseteq A_{n+1}$;
(G-iv) $|A_{n+1}| \le C_1 |A_n|$;
(G-v) *For every natural numbers n, m,*

$$\langle A_n \rangle \cap mG \subseteq m \left\langle A_{n+1+h(m)} \right\rangle .$$

Proof. We choose the A_n, C_1 and h as in Lemma 1 and prove the improved condition (G-v) for sufficiently large n depending on m. It is then easy to adjust $h(m)$ to let the condition hold for all n.

Given m, there is an n_0 such that A_{n_0} contains a representative of every element of G/mG. Note that G/mG is a finite-rank torsion-free group by Lemma 6, and so is finite by [8, Lemma 2.1.1]. Furthermore, there is an $n_1 \ge n_0$ such that $m \langle A_{n_1} \rangle$ contains the finitely generated $\langle A_{n_0} \rangle \cap mG$. For $n \ge n_1$ and $a \in A_n$, there is a $b \in A_{n_0}$ with $a + b \in mG$. Since $a + b \in A_n + A_n \subseteq A_{n+1}$, there is a $c \in A_{n+1+h(m)}$ with $mc = a + b$. This proves

$$\langle A_n \rangle \subseteq \langle A_{n_0} \rangle + m \left\langle A_{n+1+h(m)} \right\rangle$$
$$\langle A_n \rangle \cap mG \subseteq \langle A_{n_0} \rangle \cap mG + m \left\langle A_{n+1+h(m)} \right\rangle = m \left\langle A_{n+1+h(m)} \right\rangle .$$

□

Corollary 8. *For every FA-presentable torsion-free abelian group G, there is a p^* such that for all primes $p \geq p^*$, every finite subset of G is contained in a finitely generated p-pure subgroup of G.*

Proof. For a contradiction, suppose there are G and infinitely many primes for which the corollary fails. Choose $p_0, \dots, p_d$ among these satisfying (8) and (9). Let F_i be a finite set for which the prime p_i fails Lemma 7. Choose the A_n and h as in the lemma but drop the first few A_n not containing all the $F_0, \dots, F_d$. The lemma guarantees the conditions (Q-i), (Q-ii), (Q-iii) and (Q-iv) with the exception of $p_i \langle A_n \rangle \subsetneq \langle A_n \rangle \cap p_i G$. But this is also true as $\langle A_n \rangle$ containing F_i is not p_i-pure. This contradicts Proposition 4. □

Lemma 9. *For every FA-presentable torsion-free abelian group G, there is a finitely generated subgroup H such that G/H is p-divisible for all but finitely many primes p.*

Proof. First, we choose subsets A_n satisfying Lemma 7 with the notational change $C_{1,G}$ and h_G instead of C_1 and h, respectively.

Let Π denote the set of primes. We choose $C_1 := C_{1,G}^2$ and $h\colon \Pi \to \mathbb{N}$ with $h(p) := h_G(p) + 1$. Applying Proposition 5 to this C_1 and h, we obtain constants $d, K, p^* \in \mathbb{Z}^+$ and a constant $C \geq C_1$. We choose $p_0, \dots, p_d \in \Pi$ as required, i.e.

$$p_d > C(4dK)^d \text{ and } p_{i-1} > p_i C^{h(p_i)d} d^{Cd^4} \text{ for } i = d, d-1, \dots, 1.$$

Moreover, let $n_0 \in \mathbb{N}$ be such that the natural map $e_i\colon A_{n_0} \to G/p_i G$ is surjective for all $i = 0, \dots, d$. Note that n_0 exists by condition (G-i).

We claim that $\langle A_{n_0} \rangle + pG = G$ for all primes $p \geq p^*$, and hence $H := \langle A_{n_0} \rangle$ is a subgroup whose existence is claimed by the lemma. For a contradiction, suppose that there is a prime $p \geq p^*$ with $\langle A_{n_0} \rangle + pG \neq G$. Then there is a surjective $\alpha\colon G \to \mathbb{Z}/p\mathbb{Z}$ with $A_{n_0} \subseteq \ker(\alpha)$. (One first factorizes out by $\langle A_{n_0} \rangle + pG$ to obtain a non-trivial abelian group bounded by p. Such a group is a direct sum of cyclic p-groups and one projects to one of the summands.) Applying (G-i) again, there is an $n_1 > n_0$ with

$$A_{n_1-1} \subseteq \ker(\alpha),$$
$$A_{n_1} \subsetneq \ker(\alpha).$$

Put $A'_m := \alpha(A_{n_1+m})$ for $m = -1, 0, 1, \dots$, in particular $A'_{-1} = \{0\}$. We claim that conditions (F-i) to (F-iv) from Proposition 5 hold for $\langle A'_m : m \in \omega \rangle$, a contradiction. Clearly (F-i) and (F-ii) hold except for the claim $|A'_0| \leq C_1$. We now show (F-iii), which in the case $n = -1$ is just $|A'_0| \leq C_1$. Consider the epimorphism $A_{n_1+m} \xrightarrow{\alpha} A'_m$ induced by α. Then there are at least $\frac{|A_{n_1+m}|}{|A'_m|}$ elements $a_1, \dots, a_k$ that are mapped onto the same element $a \in A'_m$. Choose representatives

$b_1, \ldots, b_l \in A_{n_1+m+1}$ of A'_{m+1} and consider $\{a_i + b_j : 1 \leq i \leq k, 1 \leq j \leq l\} \subseteq A_{n_1+m} + A_{n_1+m+1} \subseteq A_{n_1+m+2}$. Since the $a_i + b_j$ are pairwise distinct,

$$\frac{|A_{n_1+m}|}{|A'_m|} \left|A'_{m+1}\right| \leq |A_{n_1+m+2}| \leq C_{1,G}^2 \, |A_{n_1+m}|$$

and hence

$$\left|A'_{m+1}\right| \leq C_{1,G}^2 \, |A'_m| \, .$$

Finally, (F-iv) follows similarly: Let $a \in A_{n_1+m}$, then, by the choice of n_0, there is $b \in A_{n_0} \subseteq A_{n_1+m}$ such that p_i divides $a + b$. Thus $a + b \in A_{n_1+m+1}$ and hence $p_i^{-1}(a + b) \in A_{n_1+m+1+h_G(p_i)}$. This finishes the proof. □

Theorem 10. *Every FA-presentable torsion-free abelian group is an extension of a finite-rank free group by a direct sum of finitely many* $\mathbb{Z}(p^\infty)$. *Especially, the FA-presentable torsion-free abelian groups of rank* 1 *are the rings* $\mathbb{Z}[1/n]$.

Proof. Let G be an FA-presentable group. Because of Lemma 6, the group G has finite rank. So by Lemma 9, there is a finitely generated H such that G/H is p-divisible by almost all primes p. As G has finite rank, we can choose H large enough so that G/H is torsion.

We show that the p-torsion part of G/H is trivial for all p except the finitely many exceptions for Corollary 8. So let p be a non-exceptional prime. Let x represent a p-torsion element in G/H. There is a finitely generated p-pure subgroup K containing H and x. So K/H is p-pure in the p-divisible group G/H, and thus itself is p-divisible. However, finitely generated p-divisible groups have trivial p-torsion parts, hence $x \in H$.

It follows that G/H is a torsion group with non-trivial p-torsion part for only finitely many primes p. Since it is a factor of a finite-rank torsion-free group, namely G, every p-torsion part has finite p-rank and so is a finite direct sum of cyclic and Prüfer groups, see [2, Exercises 3 and 6(a) of Chapter 25].

All in all, G/H is a finite direct sum of cyclic and Prüfer groups, and hence the quotient of G by a finitely generated subgroup (the preimage of the sum of cyclic components) is a finite direct sum of Prüfer groups. As the finitely generated subgroup is free, this shows that G is an extension of a finite-rank free group by a finite direct sum of Prüfer groups, as claimed. □

We conclude this paper by the following remark: By our main result every FA-presentable torsion-free abelian group G has to be an extension of a finite rank free group by a finite direct sum of Prüfer groups. However, this is still not satisfactory since there are uncountably many of such groups, e.g. take a pure finite rank subgroup of the additive group of the ring of p-adic numbers J_p. Since there are only countably many finite automata, it follows that most of these examples are not FA-presentable. Thus we have a natural question:

Question 11. Which finite rank pure subgroups of the additive group of the ring of p-adic integers are FA-presentable?

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