Independence of (co-)Hopfian p-groups

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Abstract.

1. Introduction

In [Ketao] Hopfian and co-Hopfian groups were studied and the following questions were asked:

Question 1.1.

(1) Are the classes of Hopfian and co-Hopfian p-groups the same?
(2) Is there a Hopfian or co-Hopfian p-group of size $\aleph_1$?

Our first theorem answers Question (1):

Theorem 1.2. *Given a prime p there are reduced p-groups G and H such that*

(1) G is Hopfian but not co-Hopfian; In particular, $\operatorname{End}(G) = \widehat{J_p[x]} \oplus \operatorname{small}(G)$.
(2) H is co-Hopfian but not Hopfian; In particular, $\operatorname{End}(H) = J_p[[x]] \oplus \operatorname{small}(H)$.

The next theorem takes care of Question (2):

Theorem 1.3. *The following statements are consistent with ZFC:*

(1) There is a Hopfian and co-Hopfian p-group of size $\aleph_1 < 2^{\aleph_0}$.
(2) Every reduced infinitely generated p-group of size strictly less than the continuum is neither Hopfian nor co-Hopfian.

Corollary 1.4. *The answer to Question (2) is undecidable in ZFC.*

2. A Hopfian but not co-Hopfian group

In this section, we consider the forcing extension by $\operatorname{Fin}(\omega_1, 2)$, ie, the partial poset of partial maps $\omega_1 \to \{0, 1\}$ with finite domain. We show that in this extension, there are groups of size $\aleph_1$, which are both Hopfian and co-Hopfian, or Hopfian but not co-Hopfian, or co-Hopfian but not Hopfian. Actually, it will be more convenient to consider the equivalent forcing poset $\operatorname{Fin}(\omega_1 \times \mathbb{Z}^+, 2)$. This adds new functions $\eta_\alpha \colon \mathbb{Z}^+ \to \{0, 1\}$ to the universe for $\alpha < \omega_1$.

Key words and phrases. Martin's axiom; Hopfian groups; co-Hopfian groups.

Research partially supported by Hungarian Scientific Research Fund, Grant No. NK 81203 and project STR 627/1-6 of the German Research Foundation DFG.

Let us construct some subgroups of $\prod_{n=1}^{\infty} \mathbb{Z}_{p^n} e_n$. Let x be the monomorphism defined by $e_n x := p e_{n+1}$.

$$a_{\alpha,n} := \sum_{k}^{\infty} \eta_\alpha(k) p^{k-n} e_k \tag{2.1}$$

$$a_{\alpha,n} = \eta_\alpha(n) e_n + p a_{\alpha,n+1} \tag{2.2}$$

$$G_{H,c} := \langle e_n, a_{\alpha,n} : n = 0, \ldots, \infty \rangle \tag{2.3}$$

$$G_H := \left\langle e_n, a_{\alpha,n} x^k : k, n = 0, \ldots, \infty \right\rangle \tag{2.4}$$

Lemma 2.1.

$$\operatorname{End} G_{Hc} = J_p \oplus \operatorname{small}(G_{Hc}), \tag{2.5}$$

$$\operatorname{End} G_H = \widehat{J_p[x]} \oplus \operatorname{small}(G_H). \tag{2.6}$$

Proof. Actually, we shall show the more general statement

$$\operatorname{Hom}(G_{Hc}, G_H) = \widehat{J_p[x]} \oplus \operatorname{small}(G_{Hc}, G_H). \tag{2.7}$$

In other words, for every homomorphism f and every positive integer N, there is a polynomial $k' \in \mathbb{Z}[x]$ making $f - k'$ small on $G_{Hc}[p^N]$. We are now going to prove this.

For simplicity, we abuse notation, and let f also denote the name of the homomorphism $G_{Hc} \to G_H$. There are conditions $p_{i,j}$ in the generic filter forcing the e_j-coordinate of $e_i f$ to be some $k_{i,j} \in \mathbb{Z}_{p^j}$. The $p_{i,j}$ altogether have countable support, which is therefore disjoint from $\{\alpha\} \times \omega$ for some $\alpha \in \omega_1$. There is a p in the generic filter forcing that f is a homomorphism $G_{Hc} \to G_H$, and

$$a_{\alpha,N} f = g + k a_{\alpha,M} + \sum_{i=1}^{l} k_i a_{\alpha_i,M_i} \tag{2.8}$$

for some $g \in G_H$, polynomials $k, k_i \in \mathbb{Z}[x]$ and pairwise distinct ordinals α, α_i.

For all positive integer $n \geq N$ where (α, n) is not in the support of p, let's consider the automorphism φ of the forcing notion switching the value of η_α at n: i.e., η_α^φ coincides with η_α except at the place n where $\eta_\alpha^\varphi(n) = 1 - \eta_\alpha(n)$. For $\beta \neq \alpha$, the function η_β remains unchanged: $\eta_\beta^\varphi = \eta_\beta(n)$.

On the level of partial functions, for every partial function q, we have $\operatorname{dom} q^\varphi = \operatorname{dom} q$ and $(\beta, m) q^\varphi = (\beta, m) q$ for $(\beta, m) \in \operatorname{dom} q$ and $(\beta, m) \neq (\alpha, n)$. The crucial change is $(\alpha, n) q^\varphi = 1 - (\alpha, n) q$ whenever $(\alpha, n) \in \operatorname{dom} q$. In particular, φ leaves p and the $p_{i,j}$ invariant.

Now, since p forces it, f^φ is again a homomorphism $G_{Hc} \to G_H$ satisfying

$$(a_{\alpha,N} \pm p^{n-N} e_n) f^\varphi = g + k(a_{\alpha,M} \pm p^{n-M} e_n) + \sum_{i=1}^{l} k_i a_{\alpha_i,M_i}. \tag{2.9}$$

Also, the e_j-coordinate of $e_i f^\varphi$ is $k_{i,j}$, because $p_{i,j}$ forces it, so $f^\varphi = f$. Hence the difference of the two equations (2.8) and (2.9) simplifies to

$$p^{n-N} e_n f = p^{n-M} k e_n \tag{2.10}$$

(we repeat) for all but finitely many $n \geq N$ (those for which (α, n) is not in the support of p).

It follows that $0 = p^n e_n f = p^{n+N-M} k e_n$ for almost all n. Therefore $p^{n+N-M} k$ must be divisible by p^n, which means that the polynomial $k' := p^{N-M} k$ has integral coefficients, thus (2.10) beautifies to

$$p^{n-N} e_n f = p^{n-N} k' e_n \tag{2.11}$$

for all but finitely many n.

Finally, we can conclude that $f - k'$ is zero on all but finitely many of the $p^{n-N} e_n$, hence $f - k'$ is small on $G_{Hc}[p^N]$. ∎

Proposition 2.2. *Let G be a semi-finite reduced abelian p-group with a strictly height-increasing monomorphism x, ie, $h(gx) > h(g)$ for all $g \in G$. Assume* $\operatorname{End} G = \widehat{J_p[x]} \oplus \operatorname{small}(G_H)$. *Then G is Hopfian but not co-Hopfian.*

Proof. Obviously, G is not co-Hopfian, as x is a monomorphism which is not an isomorphism. To prove G is Hopfian, let $f\colon G \to G$ be an epimorphism. We will show that it is an isomorphism adapting [**?**, 16.1–4].

First, if f is small then its image is countable. Hence G is countable, so the canonical image of $\operatorname{End} G$ in $\operatorname{Hom}(G[p], G)$ is continuum, but the image of both $\widehat{J_p[x]}$ and $\operatorname{small}(G_H)$ is only countable, eg, the first one is $\mathbb{Z}_p[x]$. So f is not small, hence is of the form $f = \alpha + \phi$ where $0 \neq \alpha \in \widehat{J_p[x]}$ and $\phi \in \operatorname{small}(G_H)$.

Second, we want to show that α has an invertible constant term in several steps.

As the first step, we decompose α as

$$\alpha = p^k x^j \beta + p^{k+1}\gamma \qquad \beta, \gamma \in \widehat{J_p[x]},\ \deg\gamma < j, \tag{2.12}$$

where β has invertible constant term. In particular, β is height-preserving, ie, $h(g\beta) = h(g)$ for all $g \in G$, hence it is a monomorphism.

The second step is determining the kernel of α:

$$\ker \alpha[p^{k+1}] = \ker p^k x^j \beta[p^{k+1}] = \ker p^k[p^{k+1}] = G[p^k] \tag{2.13}$$

using that x and β are monomorphisms, and $p^{k+1}\gamma$ is 0 on $G[p^{k+1}]$. We conclude that

$$\ker \alpha = G[p^k]. \tag{2.14}$$

Since ϕ is small, there is a natural number N such that

$$p^N G[p^{k+1}] \subseteq \ker \phi. \tag{2.15}$$

Let $n > N$ be arbitrary. As

$$p^n(G[p^n]f^{-1})[p^{k+1}] \subseteq \ker f \cap \ker \phi \subseteq \ker \alpha = G[p^k], \tag{2.16}$$

we have $p^{n-1}(G[p^n]f^{-1})[p^{k+2}] \subseteq G[p^{k+1}]$, (actually $G[p^n]f^{-1} \subseteq G[p^{k+1}]$, cf [**?**, 16.1(d)]) hence

$$p^{n-1}(G[p^n]f^{-1}) = p^{n-1}(G[p^n]f^{-1})[p^{k+1}]. \tag{2.17}$$

Consequently, cf [**?**, 16.1(b)]

$$\begin{aligned} p^{n-1}G[p] &= p^{n-1}(G[p^n]) = p^{n-1}(G[p^n]f^{-1})f = p^{n-1}(G[p^n]f^{-1})[p^{k+1}]f \\ &= p^{n-1}(G[p^n]f^{-1})[p^{k+1}]\alpha = p^{n-1}(G[p^n]f^{-1})[p^{k+1}]p^k x^j \beta \subseteq p^{n-1+k+j}G[p], \end{aligned} \tag{2.18}$$

using that ϕ and $p^{k+1}\gamma$ is 0 on $p^{n-1}(G[p^n]f^{-1})[p^{k+1}]$, and that x is strictly height-increasing.

So we have $p^{n-1}G[p] \subseteq p^{n-1+k+j}G[p]$ for all $n > N$. As G is unbounded, this is only possible if $k = j = 0$, ie, $\alpha = \beta$. Therefore α is also height-preserving.

Finally, let $B = \bigoplus_{n=1}^{\infty} B_n$ be a basic subgroup of G with B_n a direct sum of cyclic groups of order p^n. The canonical direct complement of $B_1 \oplus \dots \oplus B_N$ is $H_N \coloneqq \langle p^N G, B_{N+1}, B_{N+2}, \dots \rangle$. Now on $H_N[p] = p^N G[p]$, the small map ϕ is 0, so f coincides with α here. Hence f is height-preserving on $H_N[p] = p^N G[p]$. It follows that f is injective on H_N and $B_1 \oplus \dots \oplus B_N \oplus H_N f$ is a pure subgroup with basic subgroup isomorphic to B. Since G is semi-finite, the canonical image of $B_1 \oplus \dots \oplus B_N$

must be a (bounded) basic subgroup of $G/H_N f$, which therefore has a divisible direct complement D. Let us consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_N & \longrightarrow & G & \longrightarrow & B_1 \oplus \cdots \oplus B_N & \longrightarrow & 0 \\ & & \cong\big\downarrow & & f\big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & H_N f & \longrightarrow & G & \longrightarrow & B_1 \oplus \cdots \oplus B_N \oplus D & \longrightarrow & 0 \end{array}$$

Since f is epimorphic, the right-most vertical arrow is epimorphic. As the B_n are finite, this arrow must be an isomorphism, and $D = 0$. By the Five Lemma, therefore f is an isomorphism. □

3. A co-Hopfian but not Hopfian group

This time we use the equivalent forcing poset $\mathrm{Fin}(\omega_1 \times \mathbb{Z}^+ \times \mathbb{Z}^+, 2)$, so the newly added functions are $\eta_\alpha : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \{0, 1\}$.

$$a^{(i)}_{\alpha,n} \coloneqq \sum_{k=n+i}^{\infty} \sum_{m=1}^{k} \eta_\alpha(k-i, m) p^{k-n} e_{k,m} \qquad i \geq 0 \tag{3.1}$$

$$G_c \coloneqq \left\langle e_{n,m}, a^{(i)}_{\alpha,n} x^k : n \geq m \geq 1,\ k, i \geq 0 \right\rangle \subseteq \prod_{n=1}^{\infty} \bigoplus_{m=1}^{n} \mathbb{Z}_{p^n} e_{n,m} \tag{3.2}$$

$$e_{n,n} x \coloneqq 0 \tag{3.3}$$

$$e_{n+1,m} x \coloneqq e_{n,m} \qquad m \leq n \tag{3.4}$$

$$a^{(i+1)}_{\alpha,n+1} x = a^{(i)}_{\alpha,n} \tag{3.5}$$

$$a^{(i)}_{\alpha,n} = \sum_{m=1}^{n+i} \eta_\alpha(n, m) p^i e_{n+i,m} + p a^{(i)}_{\alpha,n+1} \tag{3.6}$$

Lemma 3.1.

$$\operatorname{End} G_c = J_p[[x]] \oplus \operatorname{small}(G_c) \tag{3.7}$$

Proof. The proof is analogous to that of Lemma 2.1, so we point out only the differences.
Now the reformulation of the statement reads: for every homomorphism f and every positive integer N, there is a polynomial $k' \in \mathbb{Z}[x]$ making $f - k'$ small on $G_c[p^N]$.
This time the forcing argument provides an α, such that

$$a_{\alpha,N} f = g + k a^{(i)}_{\alpha,M} + \sum_{j=1}^{l} k_j a^{(i_j)}_{\alpha_j, M_j} \tag{3.8}$$

for some $g \in G_c$, polynomials $k, k_i \in \mathbb{Z}[x]$, non-negative integer i and pairwise distinct ordinals α, α_j. We may assume that the constant term of k is not divisible by p^M when $i > 0$. Moreover, (still by the forcing argument via switching $\eta_\alpha(n, m)$) for all but finitely many pairs n, m of positive integers with $n \geq N$,

$$(a^{(0)}_{\alpha,N} \pm p^{n-N} e_{n,m}) f = g + k(a^{(i)}_{\alpha,M} \pm p^{n+i-M} e_{n+i,m}) + \sum_{j=1}^{l} k_j a^{(i_j)}_{\alpha_j, M_j} \tag{3.9}$$

with the convention that $e_{p,q} = 0$ for $p < q$.
The difference of the two equations (3.8) and (3.9) reduces to

$$p^{n-N} e_{n,m} f = k p^{n+i-M} e_{n+i,m} \tag{3.10}$$

(we repeat) for all but finitely many pairs n, m with $n \geq N$. If $i > 0$ then choosing $m = n + i$ and n large enough, we obtain $0 = p^{n-N} e_{n,n+i} f = k p^{n+i-M} e_{n+i,n+i}$ contradicting that the constant term of k is not divisible by p^M.
Hence $i = 0$, and we finish as in Lemma 2.1, deriving that $f - k'$ is small on $G_c[p^N]$ where $k' := p^{N-M} k$ is an integral polynomial. □

Proposition 3.2. *Let G be a semi-finite reduced abelian p-group with an endomorphism x, which is 0 on $G[p]$. Assume $\operatorname{End} G = J_p[[x]] \oplus \operatorname{small}(G_H)$. Then G is co-Hopfian (but not Hopfian if x is epimorphic).*

PROOF. To prove G is co-Hopfian, let $f \colon G \to G$ be an epimorphism. We will show that it is an isomorphism.
First, f is of the form $f = \alpha + \phi$ where $\alpha \in J_p[[x]]$ and $\phi \in \operatorname{small}(G)$.
Let n be a natural number. Since ϕ is small, there is a natural number m such that

$$p^m G[p^n] \subseteq \ker \phi, \tag{3.11}$$

so f coincides with α on $p^m G[p^n]$
Let us consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^m G[p^n] & \longrightarrow & G[p^n] & \longrightarrow & \frac{G[p^n]}{p^m G[p^n]} & \longrightarrow & 0 \\ & & \alpha \big\downarrow & & f \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & p^m G[p^n] & \longrightarrow & G[p^n] & \longrightarrow & \frac{G[p^n]}{p^m G[p^n]} & \longrightarrow & 0. \end{array}$$

Since f is monomorphic, the left-most vertical arrow α is monomorphic. Hence α must have a constant term not divisible by p, so α is invertible and therefore an isomorphism. It follows that the right-most vertical arrow is a monomorphism. As $G[p^n]/p^m G[p^n]$ is finite, this arrow must be an isomorphism. By the Five Lemma, therefore f is an isomorphism. □

4. Martin's axiom and Hopfian groups

In this section we show that MA(σ-centered) implies that every reduced p-group G of size less than $2^{\aleph_0}$ is neither Hopfian nor co-Hopfian.
We may assume that G is semi-finite, as otherwise it is neither Hopfian nor co-Hopfian. Let B be a basic subgroup of G.
We shall build endomorphisms of G by gluing together partial endomorphisms defined on finite subgroups. The partial endomorphisms will be selected via Martin's Axiom. To ensure the countable chain condition, we voluntarily restrict to endomorphisms f satisfying $G(f - 1) \subseteq B$ so that every element of G can be mapped to only countably many others. As a by-product, this also simplifies the presentation by allowing us to restrict to separable G.
The first Ulm factor $G/G^{(1)}$ is separable with a basic group isomorphic to B via the natural map $B \to G/G^{(1)}$. By abuse of notation, let B also denote this basic subgroup.
Now every endomorphism f of $G/G^{(1)}$ with image of $f - 1$ in B extends to a unique endomorphism $\widetilde{f}$ of G which is the identity on $G^{(1)}$ and satisfies $G(\widetilde{f} - 1) \subseteq B$. Hence $\widetilde{f}$ is a monomorphism or an epimorphism if and only if f is so, respectively. As f induces the identity on G/B, we derive analogously that f is monic or epic, respectively, if so is the restriction $f \colon B \to B$. In particular, f is epic if its image contains B.
Let $B = \bigoplus_{n=1}^{\infty} B_n$ be a decomposition of the basic subgroup with B_n being a direct sum of cyclic subgroups $\mathbb{Z}_{p^n}$ of order p^n. Let $H_N := \langle p^N G, B_{N+1}, B_{N+2}, \dots \rangle$ be the canonical direct complement of $\bigoplus_{n=1}^{N} B_n$ in B.

4.1. Group is not Hopfian. To show that G is not Hopfian, we apply Martin's Axiom to the following poset:

Elements: partial homomorphisms $g\colon H \to G$, where H is a finite-rank pure subgroup of G with $H(g-1) \subseteq B$.

Order: $h \leq g$ if and only if h is an extension of g.

Dense subsets:

- $\mathbb{D}_x \coloneqq \{g \mid x \in \operatorname{dom} g\}$ for $x \in G$,
- $\mathbb{D}_b \coloneqq \{g \mid x \in \operatorname{im} g\}$ for $b \in B$,
- $\mathbb{D}_{\text{non-monic}} \coloneqq \{g \mid g \text{ not monic}\}$.

Filters: $\mathcal{F}_\varphi \coloneqq \{g \mid g \subseteq \varphi\}$ for φ an endomorphism of G, which is identity on some H_n and $G(\varphi-1) \subseteq B$.

This is obviously a poset satisfying the conditions of MA(σ-centered). Eg, there are only countably many filters $\mathcal{F}_\varphi$, as there are only countably many choices of φ thanks to the restrictions $G(\varphi-1) \subseteq B$ and $\varphi \restriction H_n = 1$. These filters cover the poset, as for any partial homomorphism $g\colon H \to G$, we can find a direct complement C in G containing some H_n. Defining φ to be g on H and the identity on H_n, we obtain $g \in \mathcal{F}_\varphi$. A similar but easier argument proves that the set $\mathbb{D}_b$ is dense for $b \in B$. Now for $g\colon H \to G$, we choose a pure extension $K = H \oplus L$ with L a cyclic direct summand of B with sufficiently large order, thus g extends to an $h\colon H \to G$ with $b \in Lh \subseteq B$. It is even easier to show that the remaining sets claimed to be dense are really dense.

Applying MA(σ-centered), there is a filter intersecting the dense subsets above. Taking the union of the homomorphisms in the filter, we obtain a non-monic endomorphism f of G with $G(f-1) \subseteq B$ and $B \subseteq Gf$. It follows that f is epic but not monic, indicating that G is not Hopfian.

4.2. Group is not co-Hopfian. The construction is similar to the previous case with a suitable adjustment of the poset.

Let $0 \neq b_0 \in B$ be a fixed non-zero element of B. Our aim is to construct a monic endomorphism of G whose image does not contain b_0.

Elements: partial monomorphisms $g\colon H \to G$, where H and the image Hg are finite-rank pure subgroups of G with $b_0 \notin Hg$ and $H(g-1) \subseteq B$.

Order: $h \leq g$ if and only if h is an extension of g.

Dense subsets: $\mathbb{D}_x \coloneqq \{g \mid x \in \operatorname{dom} g\}$ for $x \in G$.

Filters: $\mathcal{F}_\varphi \coloneqq \{g \mid g \subseteq \varphi\}$ for φ a monomorphism to G defined on a subgroup of G and extending the identity on some H_n. Moreover, the image of $\varphi - 1$ lies in B and b_0 is outside the image of φ.

(For the filters, we allow φ to be defined only on a subgroup, because a priori we can't come up with a monic endomorphism of G which is not epic. Actually, the φ we use will all be restriction of an isomorphism of G.)

Since H_n is of finite index, there are only finitely many subgroups containing it, and hence there are altogether only countably many filters $\mathcal{F}_\varphi$.

To show that every partial monomorphism $g\colon H \to G$ is contained in a $\mathcal{F}_\varphi$, we proceed in several steps.

We choose decompositions $G = H \oplus C_1 \oplus H_n$ and $G = Hg \oplus C_2 \oplus H_n$ for a suitable n with $C_1, C_2 \subseteq B$. It follows that $H \oplus C_1 \cong Hg \oplus C_2$, and hence $C_1 \cong C_2$. We define φ to be g on H, an isomorphism to C_2 on C_1 and the identity on H_n. Hence φ is an automorphism of G extending g and satisfying $G(\varphi - 1) \subseteq B$. By restricting to a complement of an appropriate cyclic direct summand of $C_1 \oplus H_n$, we can omit b_0 from the image of φ with the domain still containing an H_m for some $m \geq n$. Hence $g \in \mathcal{F}_\varphi$.

An easier argument shows that the $\mathbb{D}_x$ are dense but care must be taken to omit b_0 from the image of the extension. For example, given $g\colon H \to G$, we can find decompositions $G = H \oplus C_1 \oplus H_n$ with $x \in H \oplus C_1$, and $G = Hg \oplus C_2 \oplus H_n$ with $C_2 \subseteq B$ for a suitable n. It follows that $H \oplus C_1 \cong Hg \oplus C_2$, and hence $C_1 \cong C_2$.

We define the extension $h\colon H \oplus C_1 \to G$ of g. Obviously, it should coincide with g on H. On the other part C_1, let h be a modification of the inclusion of C_1 into $G = Hg \oplus C_2 \oplus H_n$ by replacing the C_2-component with an isomorphism $C_1 \cong C_2$. Thus, $G = Hg \oplus C_1 h \oplus H_n$, hence h is an extension of g to an embedding onto a finite-rank pure subgroup. The definition was carefully made to keep the image of $h - 1$ inside B.

However, there is no guarantee that b_0 hasn't smuggled itself into the image. As a remedy, let's consider a variant h' of h: since $H_n \cap B$ is reduced and unbounded, there is an embedding $\alpha\colon C_1 \to H_n \cap B$. Now $h' := h + \alpha$ has the same properties as h: it extends g, it embeds its domain into a finite-rank pure subgroup, and the image of $h' - 1$ is in B. Moreover, $\operatorname{im} h \cap \operatorname{im} h' = Hg$, and therefore b_0 cannot be contained in the image of both h and h', so at least one of them is an extension of g in $\mathbb{D}_x$.

We conclude that the poset satisfies the conditions of MA(σ-centered). So there is a filter intersecting all the dense subsets above. The union of the monomorphisms of the filter produces a monic endomorphism f of G with $G(f-1) \subseteq B$ and $b_0 \notin Gf$. Hence f is monic but not epic, showing that G is not co-Hopfian.

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