Pathological examples of dual groups

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Contents

All groups are assumed to be Abelian !

Basic definitions

Dual groups

Let M be a group. The group $M^* = \mathrm{Hom}(M, \mathbb{Z})$ is called the dual of M. Moreover, a group A is called a dual group if it is of the form $A = M^*$ for some group M.

Example

- $A = \{0\} = A^*$
- $\mathbb{Z} = \mathbb{Z}^*$
- $\mathbb{Z}^{(\omega)} = (\mathbb{Z}^\omega)^*$
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Basic definitions

Torsionless and reflexive groups

A group M is called torsionless if the natural evaluation map

$$\sigma_M : M \to M^{**} \quad \textit{via} \quad \sigma_M(m)(\varphi) = \varphi(m)$$

from M into its double dual M^{**} is injective. Moreover, M is called reflexive if σ_M is an isomorphism.

Some results

Lemma

Let M be a group. Then

- *M is torsionless if and only if* $M \subseteq \mathbb{Z}^I$ *for some index set I;*
- M^* *is torsionless, hence dual groups are subgroups of products of copies of the integers.*

→ Dual groups are separable !

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All classical examples of infinite rank dual groups A satisfy $A \cong A \oplus \mathbb{Z}$

Theorem (Göbel, Shelah)

In various models of ZFC we can find dual groups A of infinite rank such that $A \ncong A \oplus \mathbb{Z}$.

$\rightarrow$ Are there such examples in ZFC ? OPEN PROBLEM !

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Uncountable products

Let's go back to products and free groups.....

- Given a cardinal κ there is a cardinal λ such that $(\mathbb{Z}^{\kappa})^* = \mathbb{Z}^{(\lambda)}$;
- $\mathbb{Z}^{(\lambda)}$ is reflexive if and only if λ is not ω-measurable.

Recall: λ is ω-measurable if there is a non-principal ω_1-complete ultrafilter on λ.

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Closure properties and the Reid class

The class of reflexive groups is closed under taking

- direct summands;
- direct sums over index sets of non-ω-measurable cardinality;
- products over index sets of non-ω-measurable cardinality.

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Closure properties and the Reid class

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Assume that there is no ω-measurable cardinal. Let $C(Reid)$ be the least class of non-zero groups containing $\mathbb{Z}$ and closed under direct sums and products. $C(Reid)$ is called the Reid class.

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All classically known dual groups were in the Reid class !

An example by Eda

Let $C(\mathbb{Q}, \mathbb{Z})$ be the group of continuous functions from the rationals as a topological space to $\mathbb{Z}$.

- $C(\mathbb{Q}, \mathbb{Z})$ is a pure subgroup of $\mathbb{Z}^{\mathbb{Q}}$, hence a separable group;
- $C(\mathbb{Q}, \mathbb{Z}) \cong C(\mathbb{Q}, \mathbb{Z})^{\omega}$;
- $C(\mathbb{Q}, \mathbb{Z}) \cong C(\mathbb{Q}, \mathbb{Z})^{(\omega)}$;
- $C(\mathbb{Q}, \mathbb{Z})^*$ is a non-zero dual group that is not in the Reid class;
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More *examples* of dual groups

Let's go back to the Baer-Specker group...

- Let $M = \widehat{\mathbb{Z}^{(\omega)}}$ be the closure of $\mathbb{Z}^{(\omega)}$ inside $\mathbb{Z}^{\omega}$ in the $\mathbb{Z}$-adic topology. Then $M^* = \mathbb{Z}^{(\omega)}$;
- Let λ be a cardinal and $M = \{x \in \mathbb{Z}^{(\lambda)} \mid x_j = 0$ *for all but countably many j* $\} \subseteq \mathbb{Z}^{\lambda}$. Then $M^* = \mathbb{Z}^{(\lambda)}$;
- There are continuum many subgroups of the Baer Specker group $\mathbb{Z}^{\omega}$ with dual $\mathbb{Z}^{(\omega)}$.

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More examples of dual groups

Theorem (Eda, Göbel, Paras, Pokutta, ...)

- *(MA) Every pure subgroup of $\mathbb{Z}^\omega$ of cardinality less than the continuum is a dual group;*
- *There is a dual group $A \subseteq \mathbb{Z}^\omega$ of size $\aleph_1$ which is strongly non-reflexive; in particular it is a dual group that is not reflexive.*

Definition

A group is called strongly non-reflexive if it is not isomorphic to A^{**}.

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Theorem (Mekler)

- *There is a dual group that is not of the form G^{**} for any group G;*
- *There are dual groups B and D neither of which is a double dual, but such that $B \oplus D$ is a double dual.*

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Inhalt

Problems

There is a long list of problems on dual groups - we just state three:

- (EM - Problem 12) If A is a dual group of infinite rank, is $A \cong A \oplus \mathbb{Z}$?
- (EM - Problem 8) Is there a subgroup of the Baer-Specker group that is not a dual group?
- (EM - Problem 8) Is every subgroup of the Baer-Specker group of size less than the continuum a dual group in ZFC?
- (EM - Problem 11) Is there a group A such that A^{**} is not isomorphic to H^{***} for any group H?

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Inhalt

Our answers

All of this is joint work with Gábor Braun

Theorem

There exists a pure subgroup $H \subseteq \mathbb{Z}^{\aleph_0}$ that is strongly non-reflexive but not a dual group.

An idea of the proof.....

Let

- I be an arbitrary set
- A be an ideal of non-ω-measurable subsets of I containing all finite subsets
- $G_A = \{f \in \mathbb{Z}^I | supp(f) \in A\}$

$$\Longrightarrow \quad G_A^* = G_{A^\perp} \quad and \quad G_A^{**} = G_{cl(A)}$$

where

- $A^\perp = \{u \subseteq I | \forall a \in A : | a \cap u | < \infty\}$
- $cl(A) = A^{\perp\perp} = \{u \subseteq I | \forall v \subseteq u \text{ infinite } \exists a \in A : | v \cap a | = \infty\}$.

$\Longrightarrow$ G_A is reflexive if and only if $cl(A) = A$

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Now, let $I = \mathbb{N}$ and let $\mathbb{N} = \bigcup_n H_n$ be a partition with H_n infinite. Put

$$A = \{a \subseteq \mathbb{N} :| \{n :| a \cap H_n |= \infty\} |< \infty\}$$

$\Longrightarrow$ $cl(A) = P(\mathbb{N})$

$\Longrightarrow$ G_A is not reflexive but G_A^{**} is reflexive.

$\Longrightarrow$ G_A is neither strongly reflexive nor a dual group.

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Our answers

Theorem

It is consistent with ZFC that there is a pure subgroup $A \subseteq \mathbb{Z}^{\aleph_0}$ of size $\aleph_1$ that is not a dual group and the continuum $2^{\aleph_0}$ is arbitrary large.

An idea of the proof.....

Let

- $\mathcal{P} = Fin(\omega_1 \times \omega, 2) = \{f : \omega_1 \times \omega \to 2 \mid supp(f) \text{ finite}\}$
- G a $\mathcal{P}$-generic filter
- $\mathcal{V} = V[G]$

$\longrightarrow$ $f_\alpha : \omega \to 2$ for $\alpha < \omega_1$ are added

Define

$$A = \langle S, g_\alpha : \alpha < \omega_1 \rangle_* \subseteq \prod_{i=0}^{\infty} \mathbb{Z} e_i$$

where

$$g_\alpha = \sum_i i! f_\alpha(i) e_i \quad \text{and} \quad S = \bigoplus_{i=0}^{\infty} \mathbb{Z} e_i$$

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An idea of the proof.....

CLAIM:

$$A^* = S \quad even \quad End(A) = \mathbb{Z} \oplus S \otimes A$$

PROOF: Let $f \colon A \to A$ be an endomorphism

$\to \exists\, p_{i,j} \in G$ forcing $\Phi(e_i f, e_j) = k_{i,j} \in \mathbb{Z}$ where $\Phi(e_i, e_j) = \delta_{i,j}$

$\to \exists\, \alpha \in \omega_1$ such that $\{\alpha\} \times \omega$ is disjoint from them

$\to \exists\, p \in G$ forcing $f : A \to A$, and

$$g_\alpha f = s + k g_\alpha + \sum_{i=1}^{m} k_i g_{\alpha_i}$$

for some rational numbers k, k_i and $s \in S \otimes \mathbb{Q}$.

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CLAIM:

$$A^* = S \quad even \quad End(A) = \mathbb{Z} \oplus S \otimes A$$

PROOF: Let $f\colon A \to A$ be an endomorphism

$\to \exists\, p_{i,j} \in G$ forcing $\Phi(e_i f, e_j) = k_{i,j} \in \mathbb{Z}$ where $\Phi(e_i, e_j) = \delta_{i,j}$

$\to \exists\, \alpha \in \omega_1$ such that $\{\alpha\} \times \omega$ is disjoint from them

$\to \exists\, p \in G$ forcing $f : A \to A$, and

$$g_\alpha f = s + k g_\alpha + \sum_{i=1}^{m} k_i g_{\alpha_i}$$

for some rational numbers k, k_i and $s \in S \otimes \mathbb{Q}$.

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Let $n \in \omega$ with (α, n) outside the support of p. Let

$$\varphi : \mathcal{P} \to \mathcal{P}$$

via

- $dom(q^\varphi) = dom(q)$
- $(\beta, m)q^\varphi = (\beta, m)q$ *for* $(\beta, m) \in dom(q)$, $(\beta, m) \neq (\alpha, n)$.
- $(\alpha, n)q^\varphi = 1 - (\alpha, n)q$ *if* $(\alpha, n) \in dom(q)$.

$\to$ f^φ is an endomorphism of A with

$$(g_\alpha \pm n!e_n)f^\varphi = s + k(g_\alpha \pm n!e_n) + \sum_{i=1}^{m} k_i g_{\alpha_i}$$

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Also, $\Phi(e_i f^\varphi, e_j) = k_{i,j}$ as forced by $p_{i,j}$

$\rightarrow f^\varphi = f$.

$\rightarrow e_n f = k e_n$ for all but finitely many n

$\rightarrow k \in \mathbb{Z}$ and $f - k$ is zero on all but finitely many of the e_n

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Theorem

It is consistent with ZFC that there is a pure subgroup $A \subseteq \mathbb{Z}^{\aleph_0}$ such that

$$A^{n*} \not\cong H^{(n+1)*}$$

for all $n \in \mathbb{N}$ and all groups H.

In particular, it follows from MA(σ-centered) that there is such an A preserving this property in every extension of the universe by forcing with Fin(κ, 2) for any κ.

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