Pathological examples of dual groups

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Abstract. In this note we prove three results concerning dual groups of subgroups of the Baer–Specker group. Firstly, we construct one of size $2^{\aleph_0}$, which is not a dual group, and hence strongly non-reflexive. In contrast, due to Göbel and Pokutta, under Martin's axiom every subgroup strictly smaller than the continuum is actually a dual group. Secondly, we show that the latter is not a theorem of ZFC, as adding $\aleph_1$ many Cohen reals to the ground model, we obtain a model of ZFC in which there is a non-dual subgroup of size $\aleph_1$. However, the continuum may be large. Thirdly, we construct using Martin's axiom a subgroup whose nth dual is not an $(n+1)$st dual for any n. Together our results solve two questions from the book by Eklof and Mekler [**2**].

1. Introduction

The notion of dual groups plays an important role in the theory of Abelian groups and more generally also in module theory over suitable rings. Recall that an Abelian group is a dual group if and only if it is of the form $M^* = \mathrm{Hom}(M, \mathbb{Z})$ for some Abelian group M. There is a large range of recent literature on dual modules, which can be looked up in [**2**].

Many instances of dual groups are reflexive: recall that an Abelian group A is called reflexive if the natural evaluation morphism $\sigma_A \colon A \to A^{**}$ is an isomorphism, which is defined via $\sigma_A(x)(\varphi) \coloneqq \varphi(x)$ for $x \in A$ and homomorphism $\varphi \in \mathrm{Hom}(A, \mathbb{Z})$. For example, $\mathbb{Z}$ is reflexive, as well as (non-ω-measurable) direct sums and direct products of reflexive groups. In particular, $\mathbb{Z}^{(\omega)}$ and $\mathbb{Z}^{\omega}$ are reflexive and in fact dual of each other. However, it was shown by Eda that there are reflexive groups which can not be obtained this way starting with the group of integers.

It is natural to ask whether or not all dual groups are reflexive. The answer is no, there are even non-reflexive groups A which are not isomorphic to their double duals A^{**} at all (there is also no isomorphism different from σ_A). Recall that such groups are called strongly non-reflexive. Eda and Ohta [**1**] proved the existence of a dual, strongly non-reflexive group $H \subseteq \mathbb{Z}^{\aleph_0}$ of size $\aleph_1$. (In particular, it is a dual group which is not reflexive.) We improve this result and show in Theorem 2.1 that there is even a non-dual subgroup of $H \subseteq \mathbb{Z}^{\aleph_0}$.

This already indicates that subgroups of the Baer—Specker group $\mathbb{Z}^{\aleph_0}$ are a rich source of pathological examples, which suggests that the structure of dual groups is complicated in general. For example, taking duals is by no means injective. There are continuum many subgroups of $\mathbb{Z}^{\omega}$ with $\mathbb{Z}^{(\omega)}$ as their duals with only finite rank homomorphisms between them.

This already indicates that dual groups may have small endomorphism rings. Admittedly, since they are always separable, they have many finite rank endomorphisms and decompositions of the form

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$A = A' \oplus \mathbb{Z}$. In fact, all the examples known in ZFC satisfy $A \cong A \oplus \mathbb{Z}$. Nevertheless, in the answer to Problem 12 in [**2**], three related constructions of a reflexive torsion free abelian group A are provided in [**7**], [**8**] and [**9**] such that $A \not\cong A \oplus \mathbb{Z}$. The three papers assume additional set-theoretic assumptions, namely the diamond principle for $\aleph_1$, the special continuum hypothesis CH and Martin's axiom, respectively. In fact, in [**8**] a marginally weaker assumption, namely the σ-centered version of Martin's axiom was sufficient. If we strengthen the separability condition, then assuming Shelah's proper forcing axiom every $\aleph_1$-separable group of cardinality $\aleph_1$ is reflexive. On the other hand, in Gödel's constructible universe there is an $\aleph_1$-separable group of size $\aleph_1$ which is not reflexive. Finally, it has been proved recently by Pokutta and Göbel that under Martin's axiom every pure subgroup A of $\mathbb{Z}^\omega$ of cardinality less than the continuum is a dual group.

One of our main theorems (Theorem 2.2) states that in contrast to the result by Göbel and Pokutta there is a model of ZFC that allows a pure subgroup of the Baer–Specker group of size $\aleph_1$, which is not a dual group, and the continuum is large.

Our final result, Theorem 2.3 improves the previous constructions under Martin's axiom to construct a group whose n^{th} dual is not an $(n+1)^{\text{st}}$ dual for any n, answering Question 11 on dual groups of [**2**].

2. Main results

In this section we summarize our results on dual groups of Abelian groups, which will be proved in the rest of the paper. These shed light on some of the open problems in [**2**].

In this paper all groups will be Abelian and refer to the standard book by Fuchs [**4, 3**] for Abelian group theory. Moreover, the paper requires a basic knowledge on forcing and set theory as developed in Kunen's book [**10**].

The group $G^{**} = (G^*)^*$ is called the *double dual* of G and similarly taking the dual n times we call G^{n*} the n^{th} dual of G.

We begin by constructing a non-dual subgroup the Baer–Specker group, complementing the above mentioned construction of Eda and Ohta [**1**] of a dual, strongly non-reflexive subgroup.

Theorem 2.1 (Question 8 on dual groups in [**2**]). *There exists a pure subgroup $H \subseteq \mathbb{Z}^{\aleph_0}$ that is strongly non-reflexive but not a dual group.*

The group in the above theorem has size continuum. This is justified by the following: We will show that the existence of pure non-dual subgroups of the Baer—Specker group of size less than continuum is undecidable in ZFC. On the one hand, recall that MA(σ-centered), a weak form of Martin's Axiom, implies that there is no such subgroup ([**6**, Theorem 4.13]). On the other hand, we show that by adding $\aleph_1$ many Cohen reals via forcing with $\mathrm{Fin}(\omega_1, 2)$, we obtain pure non-dual subgroups of size $\aleph_1$. Recall that $\mathrm{Fin}(X, Y)$ is the poset of partial functions $X \to Y$ with finite support.

Theorem 2.2. *It is consistent with ZFC that there is a pure subgroup $A \subseteq \mathbb{Z}^{\aleph_0}$ of size $\aleph_1$ that is not a dual group and the continuum $2^{\aleph_0}$ is arbitrary large. In particular, forcing with $\mathrm{Fin}(\omega_1, 2)$, there is such an A.*

Recall that MA(σ-centered) was successfully used to construct pathological reflexive groups, e.g., [**8**, Theorem 1.2] and [**5**, Theorem 1.1]. The above theorem might suggest that forcing with $\mathrm{Fin}(\kappa, 2)$ for some large κ gets rid of these examples, but this is not the case. We show this by solving another open question on dual groups and suitably modifying the above mentioned constructions.

Theorem 2.3 (Question 11 on dual groups in [**2**]). *It is consistent with ZFC that there is a pure subgroup $A \subseteq \mathbb{Z}^{\aleph_0}$ such that for all $n \in \mathbb{N}$ and all groups H we have $A^{n*} \ncong H^{(n+1)*}$. In particular, it follows from MA(σ-centered) that there is such an A preserving this property in every extension of the universe by forcing with* $\mathrm{Fin}(\kappa, 2)$ *for any κ.*

3. A non-dual strongly non-reflexive group - Theorem 2.1

We now prove Theorem 2.1 using a construction of Shelah from [**12**], which we recall before the proof for the reader's convenience.
Let I be an arbitrary set and let A be an ideal of non-ω-measurable subsets of I containing all finite subsets. (See [**2**, Definition 2.10] for definition of ω-measurable). Define the group

$$G_A := \{f \in \mathbb{Z}^I \,|\, \operatorname{supp} f \in A\} \tag{3.1}$$

where $\operatorname{supp} f = \{i \in I | f(i) \neq 0\}$. Its dual group is easy to describe:

$$G_A^* = G_{A^\perp} \tag{3.2}$$

with the definition

$$A^\perp := \{u \subseteq I | \forall a \in A : |a \cap u| < \infty\}. \tag{3.3}$$

The double dual has a similar description (provided $A^\perp$ contains no ω-measurable set):

$$G_A^{**} = G_{\mathrm{cl}(A)} \tag{3.4}$$

with

$$\mathrm{cl}(A) := A^{\perp\perp} = \{u \subseteq I | \forall v \subseteq u \text{ infinite } \exists a \in A : |v \cap a| = \infty\}. \tag{3.5}$$

In particular, G_A is reflexive if and only if $\mathrm{cl}(A) = A$. Note that $G_{\mathrm{cl}(A)}$ and $G_{A^\perp}$ are always reflexive (provided $A^\perp$ contains no ω-measurable set).
We are ready for the application.

Proof of Theorem 2.1. We apply the above construction in the special case $I = \mathbb{N}$. First, partition $\mathbb{N} = \bigcup_n H_n$ into infinitely many infinite sets H_n. Let A be the ideal of sets intersecting all but finitely many partition H_n in a finite set:

$$A := \{a \subseteq \mathbb{N} : |\{n : |a \cap H_n| = \infty\}| < \infty\} \tag{3.6}$$

Then $\mathrm{cl}(A)$ is the set of all subsets of $\mathbb{N}$, hence G_A is not reflexive but G_A^{**} is reflexive. It follows that G_A is not isomorphic to a direct summand of its double dual G_A^{**}, hence it is neither strongly reflexive nor a dual group. □

4. Pure non-dual subgroups of $\mathbb{Z}^\omega$ with $\aleph_1$ Cohen real numbers - Theorem 2.2

Proof of Theorem 2.2. For convenience, instead of $\mathrm{Fin}(\omega_1, 2)$, we shall use the isomorphic forcing notion $\mathrm{Fin}(\omega_1 \times \omega, 2)$ i.e., the poset of functions $\omega_1 \times \omega \to 2$ with finite support. This adds a new function $\omega_1 \times \omega \to 2$ to the ground model, which we break up to functions $f_\alpha : \omega \to 2$ for $\alpha < \omega_1$ for convenience. With the help of these functions, we define elements of the Baer—Specker group $\prod_{i=0}^\infty \mathbb{Z} e_i$:

$$g_\alpha := \sum_i i! f_\alpha(i) e_i. \tag{4.1}$$

These are the generators for our pure non-dual subgroup together with the free subgroup

$$S \coloneqq \bigoplus_{i=0}^{\infty} \mathbb{Z} e_i, \tag{4.2}$$

so we define

$$A \coloneqq \langle S, g_\alpha : \alpha < \omega_1 \rangle_*. \tag{4.3}$$

We will show $A^* = S$ given by the pairing Φ defined via $\Phi(e_i, e_j) = \delta_{i,j}$. More generally, $\operatorname{End} A = \mathbb{Z} \oplus S \otimes A$, i.e., every endomorphism is a sum of a scalar multiplication and a finite-rank morphism. Let G denote the generic filter.

Let $f \colon A \to A$ be an endomorphism. There are $p_{i,j} \in G$ forcing $\Phi(e_i f, e_j) = k_{i,j}$ for some integers $k_{i,j}$. These together have countable support, so there is an $\alpha \in \omega_1$ such that $\{\alpha\} \times \omega$ is disjoint from them. There is a $p \in G$ forcing f to be an endomorphism $A \to A$, and

$$g_\alpha f = s + k g_\alpha + \sum_{i=1}^{m} k_i g_{\alpha_i} \tag{4.4}$$

for some rational numbers k, k_i and $s \in S \otimes \mathbb{Q}$.

Let $n \in \omega$ with (α, n) outside the support of p. Consider the automorphism φ of the forcing notion switching the value at (α, n): i.e., for every partial function q, we set $\operatorname{dom} q^\varphi = \operatorname{dom} q$ and $(\beta, m) q^\varphi = (\beta, m) q$ for $(\beta, m) \in \operatorname{dom} q$ and $(\beta, m) \neq (\alpha, n)$. Finally, $(\alpha, n) q^\varphi = 1 - (\alpha, n) q$ if $(\alpha, n) \in \operatorname{dom} q$. In particular, φ leaves p and the $p_{i,j}$ invariant.

Then f^φ is an endomorphism of A with

$$(g_\alpha \pm n! e_n) f^\varphi = s + k(g_\alpha \pm n! e_n) + \sum_{i=1}^{m} k_i g_{\alpha_i}, \tag{4.5}$$

because this is forced by p.

Also, $\Phi(e_i f^\varphi, e_j) = k_{i,j}$ as forced by $p_{i,j}$, so $f^\varphi = f$. Hence the difference of the two equations (4.5) and (4.4) simplifies to

$$e_n f = k e_n \tag{4.6}$$

for all but finitely many n (those for which (α, n) is in the support of p). Hence k is an integer, and $f - k$ is zero on all but finitely many of the e_n, so $f - k \in S \otimes A$ as claimed. □

5. Reflexive subgroups under MA(σ-centered) - Theorem 2.3

We will prove Theorem 2.3, i.e., construct a group A whose n^{th} dual is not an $(n+1)^{st}$ dual.

The duality between groups shall come from bilinear maps $\Phi \colon A \times B \to \mathbb{Z}$. We introduce a notion that Φ locally induces a duality.

Definition 5.1. A bilinear map $\Phi \colon A \times B \to \mathbb{Z}$ between torsion-free abelian groups is *left dense* if it satisfies the following equivalent conditions:

(1) The homomorphism $A \to B^*$ induced by Φ has a dense subgroup as image in the $\mathbb{Z}$-adic topology.
(2) The homomorphism $B \to A^*$ induced by Φ is a pure embedding.
(3) For every finite rank pure subgroup B_1 of B, there is a finite rank pure subgroup A_1 of A such that Φ induces isomorphisms $A_1 \cong B_1^*$ and $B_1 \cong A_1^*$. (In particular, A_1 and B_1 are free direct summands.)

One can similarly define right dense bilinear maps. A bilinear map is *dense* if it is both left and right dense.

PROOF THAT THE CONDITIONS ARE EQUIVALENT. (1) $\implies$ (3). Let B_1 be a finite-rank pure subgroup of B, hence its dual is a finite-rank free group. By density, the homomorphism $A \to B_1^*$ induced by Φ is surjective, thus it splits: there is a subgroup A_1 of A for which Φ induces an isomorphism $A_1 \cong B_1^*$.
(3) $\implies$ (2). By assumption, for every finite-rank pure subgroup B_1 of B, there is a subgroup A_1 of A such that the composition of the induced map $B \to A^*$ with the restriction $A^* \to A_1^*$ is an isomorphism. Hence the map $B \to A^*$ is a split monomorphism, and therefore also a pure embedding. It follows that the map $B \to A^*$ is a pure embedding.
(2) $\implies$ (1). For density, we show that Φ induces an onto map $A \to B_1^*$ for every finite-rank pure subgroup B_1 of B. Let K be the kernel of the induced map $A \to B_1^*$, so the rank of A/K is not larger than the rank of B_1. Furthermore, B_1 embeds into A^* via $B \to A^*$ as a pure subgroup. Actually, it embeds into the subgroup $(A/K)^*$ as a pure subgroup. Since the rank of B_1 is at least the rank of A/K, this is only possible if B_1 embeds onto $(A/K)^*$, i.e., $B_1 \cong (A/K)^*$ via Φ. It follows that $A/K \cong B_1^*$ via Φ. (This argument extends to show (3) $\implies$ (1).) □

5.1. Setup and notation. We shall introduce frequently used notations. These definitions are always interpreted in the ground model.
Let $\overline{X}$ denote the $\mathbb{Z}$-adic closure of X. Let $S := \mathbb{Z}^{(\omega)}$ be an infinitely-countable-rank free group with a bilinear map $\Phi\colon S \times S \to \mathbb{Z}$. This extends via continuity to a bilinear map $\widehat{S} \times \widehat{S} \to \widehat{\mathbb{Z}}$ between the $\mathbb{Z}$-adic completions. For a subgroup $H \subseteq \widehat{S}$ of the right argument, let $\mathbb{D}(H)$ be the $\mathbb{Z}$-adic closure of S in H^*:

$$\mathbb{D}(H) = \widehat{S} \cap H^* := \{x \in \mathbb{D} : \Phi(x, H) \subseteq \mathbb{Z}\}. \tag{5.1}$$

Here $\widehat{S} \cap H^*$ is meant as a mnemonic rather than a precise formula. We shall use the short-hand $\mathbb{D} := \mathbb{D}(S)$ for the $\mathbb{Z}$-adic closure of S in $\mathbb{Z}^\omega = S^*$.
We will decompose the left argument S into the direct sum of two free groups S_A and S_R of countable rank: $S = S_A \oplus S_R$. We shall use the above definitions also for the restriction of Φ to $S_A \times S$ and $S_R \times S$ by adding an A or R, respectively, to the notation: so we have $\mathbb{D}_A$, $\mathbb{D}_A(H)$ and $\mathbb{D}_R$, $\mathbb{D}_R(H)$.

5.2. Iterated dual groups. In this subsection, we reduce Theorem 2.3 to the following.

Proposition 5.2. *Assuming MA(σ-centered), there is a subgroup A of the Baer—Specker group such that in every extension of the universe by forcing with* $\mathrm{Fin}(\kappa, 2)$ *for any κ,*

1. *$A^{**} = A \oplus R$ where R is a non-zero reflexive group not isomorphic to $R \oplus \mathbb{Z}^k$ for any positive integer k.*
2. *The kernel of the induced epimorphism $A^* \to R^*$ is countable.*
3. $\mathrm{End}(A \oplus R) = \mathbb{Z}1_A \oplus \mathbb{Z}1_R \oplus (A^* \oplus A^*/K) \otimes (A \oplus R)$*. In other words, the endomorphisms of $A \oplus R$ are the scalar multiplications componentwise, the finite-rank endomorphisms and sums of these.*

The A in the proposition satisfies Theorem 2.3, which is the content of the next lemma.

Lemma 5.3. *Let A be an abelian group whose natural homomorphism $A \to A^{**}$ is a split monomorphism with direct complement a non-zero reflexive group R. Let K be the kernel of the induced epimorphism $f\colon A^* \to R^*$, and let it be countable. Furthermore, let's assume* $\mathrm{End}(A \oplus R) = \mathbb{Z}1_A \oplus \mathbb{Z}1_R \oplus (A^* \oplus A^*/K) \otimes (A \oplus R)$.

Then n^{th} dual A^{n*} of A is not an $(n+1)^{st}$ dual $H^{(n+1)*}$ for any n and group H.

Proof. It is enough to prove this for even numbers, i.e., $A^{(2n)*}$ is not of the form $H^{(2n+1)*}$. Let us assume for a contradiction that

$$H^{(2n+1)*} = A^{(2n)*} = A \oplus R^n. \tag{5.2}$$

Now H^* is naturally a direct summand of $H^{(2n+1)*}$, so

$$H^* \oplus \mathbb{Z}^k \cong A^i \oplus R^j \oplus \mathbb{Z}^l \tag{5.3}$$

as $A \oplus R^n$ has only the obvious projections by assumption.
Taking the $2n^{\text{th}}$ dual of both sides we obtain

$$A \oplus R^n \oplus \mathbb{Z}^k = H^{(2n+1)*} \oplus \mathbb{Z}^k \cong A^i \oplus R^{ni+j} \oplus \mathbb{Z}^l. \tag{5.4}$$

Again, because of the few endomorphisms of $A \oplus R$, this is only possible in the obvious case $i = 1$, $ni + j = n$, $k = l$, hence $j = 0$ and $H^* \cong A$.
The isomorphism $H^* \cong A$ induces a homomorphism $\varphi\colon H \to A^*$. Composing with the map $f\colon A^* \to R^*$, we obtain a map $H \to R^*$, which has finite rank, because its dual map $R \to H^* \cong A$ has finite rank, as every map $R \to A$ does.
Since the kernel of f is countable and the image of φf has finite rank, the image of φ is countable and hence free. Thus $A = (\operatorname{im} \varphi)^*$ is reflexive, contradicting $R \neq 0$. □

5.3. The role of MA(σ-centered). Former constructions for dual groups, [**5**, Main Lemma 2.1] and [**6**, Step Lemmas 3.1, 4.10, and 5.2], used MA(σ-centered) to create various elements of $\mathbb{D}(H)$, i.e., showing that it is large. We formulate a common generalization of these to isolate the use of MA.

Lemma 5.4. *Let $\Phi\colon S \times S \to \mathbb{Z}$ be a bilinear map such that the induced map $S \to S^*$ from its right argument to the dual of its left argument has a pure image (e.g., Φ is left dense). Let H be a pure subgroup of $\mathbb{D}$ containing S of size less than $2^{\aleph_0}$. Then every homomorphism $\varphi\colon \mathbb{D}(H) \to T$ to a slender group T is a finite sum $\sum_i \Phi(-, h_i)t_i$, where $h_i \in H$ and $t_i \in T$.*

As an obvious corollary we recover [**6**, Main Theorem 4.13] partly, namely, the H in the lemma are dual groups. This has the benefit of obtaining an explicit, easily definable primal group of H, but it does not produce a rigid system of primal groups as in [**11**, Theorem 1.5.8].

Corollary 5.5. *Every pure subgroup $S \subseteq_* H \subseteq_* \mathbb{D}$ of size less than $2^{\aleph_0}$ is a dual group. In particular, $H = \mathbb{D}(H)^*$.*

Proof of Lemma 5.4. Let K be the kernel of the induced map $S \to S^*$. By replacing Φ with the induced bilinear map $S \times S/K \to \mathbb{Z}$ and H with H/K, we may assume the induced map $S \to S^*$ is injective, and hence Φ is left dense. The main claim is that there are finitely many elements $h_1, \dots, h_n \in H$ with $\bigcap_{i=1}^n \ker \Phi(-, h_i) \nsubseteq \ker \varphi$. Assuming it for the moment, we show that φ is of the claimed form. Note that the conditions on H ensure that $\Phi\colon \mathbb{D}(H) \times H \to \mathbb{Z}$ is also left dense. First we modify the h_i so that they form a basis of a direct summand, e.g., by replacing them with a basis of the pure subgroup $\langle h_1, \dots, h_n \rangle_*$ generated by them. Hence the map $\mathbb{D}(H) \to \mathbb{Z}^n$ with components $\Phi(-, h_1), \Phi(-, h_2), \dots, \Phi(-, h_n)$ is surjective by Definition 5.1(1), so by assumption φ factors through it. This means that $\varphi = \sum_{i=1}^n \Phi(-, h_i)t_i$ where the t_i are the images of the canonical basis of $\mathbb{Z}$ in T.
Returning to the main claim, for a contradiction, let us suppose that $\bigcap_{i=1}^n \ker \Phi(-, h_i) \nsubseteq \ker \varphi$ for any finite subfamily $h_1, \dots, h_n$ of H. We will show that there exists a homomorphism $f\colon \mathbb{D} \to \mathbb{D}(H)$, whose composition $f\varphi$ with φ maps every e_i to non-zero, contradicting the slenderness of T.

For a moment, let us allow f to have codomain $\widehat{S}$. We can arbitrarily choose the images $a_i = e_i f$, and this will uniquely determine f. We only need to take care of $a_i \varphi \neq 0$.
To actually have the image in $\mathbb{D}(H)$, the condition is that for every $h \in H$, for all but finitely many i depending on h, we have $\Phi(e_i f, h) = 0$.
All in all, we need to find elements $a_i \in \mathbb{D}(H)$ with $a_i \varphi \neq 0$ but $\Phi(e_i f, h) = 0$ for every $h \in H$ and almost all $i \in I$ depending on h.
We will use Martin's axiom to choose the a_i. Actually, all the a_i will lie in S. Heuristically, the forcing conditions have to decide not just the a_i, but also for every $h \in H$ the finitely many i for which $\Phi(a_i, h) = 0$. A plausible choice for the forcing conditions are to determine finitely many of these objects. This motivates the following definition of the forcing poset:

Elements: $(a_1, \dots, a_n; U)$ with $a_i \in \mathbb{D}(H) \setminus \ker \varphi$ and U a finite subset of H. (Heuristically: an initial segment of the $\{a_i\}$ with $a_i h = 0$ for $h \in U$ and $i > n$.)

Order: $(a_1, \dots, a_n; U) \leq (b_1, \dots, b_m; V)$ if
(1) $n \leq m$
(2) $b_i = a_i$ for $1 \leq i \leq n$
(3) $U \subseteq V$
(4) $b_j h = 0$ for $n < j \leq m$ and $h \in U$.

Filters: $\{(a_1, \dots, a_n; U) : U \subseteq H \text{ finite}\}$ for a sequence $a_1, \dots, a_n$.

Dense subsets: Heuristically, these ensure that everything is decided which must be.
- $D_m := \{(a_1, \dots, a_n; U) : n \geq m\}$ for positive integers m,
- $D_h := \{(a_1, \dots, a_n; U) : h \in U\}$ for $h \in H$.
(These are dense because $\bigcap_{h \in U} \ker \Phi(-, h) \nsubseteq \ker \varphi$.)

The countably many filters above cover the poset, showing that it is σ-centered. The number of dense subsets above are less than $2^{\aleph_0}$ because the size of H is less than $2^{\aleph_0}$. Hence MA(σ-centered) applies, and there is a filter $\mathcal{F}$ intersecting all the dense subsets above.
The sequences $a_1, \dots, a_n$ in the elements of $\mathcal{F}$ are compatible and hence combine to a sequence $a_1, \dots, a_n, \dots$, which is infinite, as the dense subsets D_m take care of it. Finally, for every $h \in H$, the filter contains an element $(a_1, \dots, a_n; U)$ of the dense subset D_h, i.e., $h \in U$. Then $\varphi(a_i, h) = 0$ for $i > n$. □

5.4. Adding just one Cohen real number. Throughout this subsection, we are working in a forcing extension by $\mathrm{Fin}(\omega, 2)$ of a ground model for some simplification. In Subsection 5.5, we will see that the present construction works for $\mathrm{Fin}(\kappa, 2)$ for all κ simultaneously. We are still assuming MA(σ-centered) in the ground model.
The next lemma extends Lemma 5.4 to the extension model, while still using the groups of the ground model.

Lemma 5.6. *Let $\Phi \colon S \times S \to \mathbb{Z}$ be a bilinear map. In the ground model, let H be a pure subgroup of $\mathbb{D}$ containing S with cardinality less than $2^{\aleph_0}$ and let T be any group. Then every homomorphism $f \colon \mathbb{D}(H) \to T$ in the extension model lies in fact in the ground model. In particular, if T is reduced torsion-free, and the map $S \to S^*$ induced by Φ from its right argument to the dual of its left argument has a pure image (e.g., Φ is left dense), then f is a finite sum $\sum_i \Phi(-, h_i) t_i$ for some $h_i \in H$ and $t_i \in T$.*

Proof. The second claim follows from the first one by Lemma 5.4. Thus we only show that f lies in the ground model.

The idea is to construct a single $x \in \mathbb{D}(H)$, for which a condition can decide $xf \in T$ only if it almost completely decides f, and hence forces $f \in V$.
We construct x via MA(σ-centered) in the ground model. Now we are dealing with two posets at the same time: the poset used to construct x, and the poset $\operatorname{Fin}(\omega, 2)$ used for constructing the extension model. For clarity, we use forcing notions for the former only in the heuristic motivating its definition, and otherwise reserve forcing notions for the latter.
Heuristically, the elements of the former poset decide finitely many of the objects $x \mod n$ for $n \in \mathbb{N}$ and $\Phi(x, h) \in \mathbb{Z}$ for $h \in H$. If finitely many such decisions are compatible, then there is also an $x = a \in S$ satisfying these. This is not only to simplify the definition of the poset, but for covering the poset with countably many filters corresponding to the $x \in S$ case.
The actual poset, motivated by the heuristic, is:

Elements: (a, n, F) with $a \in S$, $n \in \mathbb{Z}^+$, $F \subseteq H$ finite subset. (Heuristically: (a, n, F) forces $x \equiv a \mod n$ and $\Phi(x, h) = \Phi(a, h)$ for all $h \in F$.)

Preorder: $(b, m, K) \leq (a, n, F)$ if
(1) $n \mid m$
(2) $n \mid b - a$
(3) $F \subseteq K$
(4) $\Phi(a, h) = \Phi(b, h)$ for all $h \in F$.

Filters: $\mathcal{F}_a \coloneqq \{(a, n, F) : n \in \mathbb{Z}^+, F\}$
(1) $D_k \coloneqq \{(a, n, F) : k \mid n\}$ for $k \in \mathbb{Z}^+$
(2) $D_h \coloneqq \{(a, n, F) : h \in F\}$ for $h \in H$
(3) $D_p \coloneqq \{(a, n, F) : p$ does not decide $af \mod n\}$ for $p \in \operatorname{Fin}(\omega, 2)$ a condition not forcing f to be in the ground model.

We show that the D_p are dense. Let us assume that nothing below (a, n, F) is in D_p, and we will show that p forces $f \in V$. For a finite subset $F \subseteq H$, let $\ker F$ denote the subgroup of S consisting of the elements where all the homomorphisms in F are zero. Obviously, S is generated by $\ker F$ and finitely many elements.
By assumption for all $b \in \ker F$ and $n \mid m$, we have $(a + nb, m, F) \notin D_p$, thus p decides $(a + nb)f$ modulo m. Fixing b and varying m, we obtain that p decides $(a + nb)f$, as T is reduced. Thus p decides f on $a + n \ker F$, and so forces the restriction of f to this coset to be in the ground model V.
We show that $f \restriction (a + n \ker F) \in V$ implies $f \in V$. Because T is torsion-free, $f \restriction (a + n \ker F)$ decides f on $\ker F$. As S is generated by $\ker F$ and finitely many elements, it is easy to recover f on S in V. Since T is reduced, the extension of f from S to $\mathbb{D}(H)$ is unique and definable in V, so f is in V.
Hence applying MA(σ-centered), there exists a filter $\mathcal{F}$ intersecting the above mentioned dense sets. We define x of the $\mathbb{Z}$-adic completion of S via

$$x \equiv a \pmod{n}, \quad (a, n, F) \in \mathcal{F}. \tag{5.5}$$

Since $\mathcal{F}$ intersects the D_k, the element x is uniquely determined. From the definition of the partial order, it is obvious that

$$\Phi(x, h) = \Phi(a, h), \quad (a, n, F) \in \mathcal{F}, h \in F. \tag{5.6}$$

Since $\mathcal{F}$ intersects the D_h, it follows that $x \in \mathbb{D}(H)$. So there must be a forcing condition p in the generic filter forcing $xf = t$ for some $t \in T$. Then p forces

$$xf \equiv af = t \pmod{n} \tag{5.7}$$

for every $(a, n, F) \in \mathcal{F}$, i.e., $\mathcal{F}$ does not intersect D_p. Hence p forces $f \in V$, so this is indeed the case. □

We are ready to formulate our homomorphism killing lemmas. Thanks to the previous lemma, we do not have to bother directly with set theoretic assumptions.

Lemma 5.7. *Let $\Phi\colon S \times S \to \mathbb{Z}$ be a bilinear map, such that the image of the left argument is pure in S^*. Let $S \subseteq_* A, B \subseteq_* \mathbb{D}$ be pure subgroups of size less than $2^{\aleph_0}$ in the ground model with $B \cap \overline{S_A} = S_A$ and $\Phi(A, B) \subseteq \mathbb{Z}$. Let f be a homomorphism $S \to \mathbb{Z}$ in the extension model. Then either $f = \Phi(a, -)$ for some $a \in A$ or there is a pure extension $B' = \langle B, b \rangle_*$ in $\mathbb{D}$ inheriting the properties $B' \cap \overline{S_A} = S_A$ and $\Phi(A, B') \subseteq \mathbb{Z}$ from B, but furthermore, $bf \nsubseteq \mathbb{Z}$.*

Proof. Actually, in this proof we are using the bilinear map Φ^{opp} instead of Φ, i.e., we exchange the arguments. In particular, the groups $\mathbb{D}(A)$ etc are defined with respect to Φ^{opp}.
Let us suppose that there is no such B'. Let $b \in \mathbb{D}(A)$ and $B' := \langle B, b \rangle_*$. Then we have $\Phi(A, B') \subseteq \mathbb{Z}$, so by assumption, either $bf \notin \mathbb{Z}$ or $B' \cap \overline{S_A}$ has become strictly larger than S_A. In the latter case, we must have $b \in B + \overline{S_A}$. All in all, the assumption implies

$$\mathbb{D}(A) \subseteq (B + \overline{S_A}) \cup \mathbb{Z}f^{-1}. \tag{5.8}$$

Recall that the only way a subgroup H can be covered by two others H_1 and H_2 is that H is contained in either H_1 or H_2. Hence $\mathbb{D}(A)$ is contained in either $(B + \overline{S_A})$ or $\mathbb{Z}f^{-1}$. Note that

$$\mathbb{D}_R(A) \cap (B + \overline{S_A}) = \frac{\mathbb{D}_R(A) \cap (B + \overline{S_A})}{\mathbb{D}_R(A) \cap \overline{S_A}} \cong \frac{B \cap (\mathbb{D}_R(A) + \overline{S_A})}{B \cap \overline{S_A}}. \tag{5.9}$$

Thus, the torsion-free group $\mathbb{D}_R(A) \cap (B + \overline{S_A})$ is a subquotient of B, so it has size less than $2^{\aleph_0}$, and hence it is slender. But $\mathbb{D}_R(A)$ is not slender by Lemma 5.4 applied to $\Phi^{\mathrm{opp}} \restriction S_R \times S$. (It does not matter whether the image of A in S_R^* is pure, because $\mathbb{D}_R(A) = \mathbb{D}_R(\widetilde{A})$ where $\widetilde{A}$ is the purification of he image of A, so we can replace A by $\widetilde{A}$.) Therefore $\mathbb{D}_R(A)$ is not contained in $B + \overline{S_A}$. Hence the only possibility for (5.8) is $\mathbb{D}(A) \subseteq \mathbb{Z}f^{-1}$, i.e., $\mathbb{D}(A)f \subseteq \mathbb{Z}$. Lemma 5.6 applies to show that f must be of the form $\Phi(a, -)$ for some $a \in A$. □

A problem with the previous lemma is that it is hard to iterate in the ground model, as it does not see the homomorphisms in the extension model. It sees only names of the homomorphisms. So we modify the lemma to kill all possible representatives of a name in one shot.

Lemma 5.8. *Let $\Phi\colon S \times S \to \mathbb{Z}$ be a bilinear map, such that the image of the left argument is pure in S^*. Let $S \subseteq_* A, B \subseteq_* \mathbb{D}$ be pure subgroups of size less than $2^{\aleph_0}$ in the ground model with $B \cap \overline{S_A} = S_A$ and $\Phi(A, B) \subseteq \mathbb{Z}$. Let f be a name of a homomorphism $S \to \mathbb{Z}$ in the extension model. Then there is a pure extension B' of B in $\mathbb{D}$ by a countable group in the ground model inheriting the properties $B' \cap \overline{S_A} = S_A$ and $\Phi(A, B') \subseteq \mathbb{Z}$ from B, but furthermore, it is forced that either $f = \Phi(a, -)$ for some $a \in A$ or $B'f \nsubseteq \mathbb{Z}$.*

Proof. Choose a maximal family of pairs p_i, b_i such that p_i forces $b_i f \notin \mathbb{Z}$, the p_i are pairwise incompatible, and $B' := \langle B, b_i : i \rangle_*$ inherits $B' \cap \overline{S_A} = S_A$ and $\Phi(A, B') \subseteq \mathbb{Z}$. By the countable chain condition, the family is countable so B' is countably generated over B. By Lemma 5.7, it is forced that either f is $\Phi(a, -)$ for an $a \in A$, or there is a b with $B'' := \langle B', b \rangle_*$ inheriting $B'' \cap \overline{S_A} = S_A$ and $\Phi(A, B'') \subseteq \mathbb{Z}$, but $bf \notin \mathbb{Z}$. We show that in the latter case $B'f \nsubseteq \mathbb{Z}$ finishing the proof. Suppose for a contradiction, that a p forces the above conditions for some b and also $B'f \subseteq \mathbb{Z}$. Hence p is incompatible with all the p_i, thus p, b can be added to the family p_i, b_i, contradicting its maximality. □

So far we have killed homomorphisms to $\mathbb{Z}$. Now we are going to kill endomorphisms. We formulate a pair of lemmas similar to above: first we kill an explicit homomorphism, then we generalize to kill a name of a homomorphism.

Lemma 5.9. *Let $\Phi\colon S \times S \to \mathbb{Z}$ be a bilinear map, such that the image of the right argument S is pure under the induced maps $S \to S_A^*$ and $S \to S_R^*$. In the ground model, let B, A, and R be pure subgroups of size less than $2^{\aleph_0}$ of $\mathbb{D}$, $\mathbb{D}_A$, and $\mathbb{D}_R$, containing S, S_A, and S_R, respectively. Furthermore, assume $B \cap \overline{S_A} = S_A$. Let f be a homomorphism $S \to \mathbb{D}$ in the extension model. Then either*

(1) $f \restriction A = k_A + \sum_j \Phi(-, b_j)x_j$ for some $k_A \in \mathbb{Z}$, $b_j \in B$, $x_j, \in A \oplus R$; or

(2) there are pure extensions $B' = \langle B, b\rangle_*$, and $A' = \langle A, a\rangle_*$ in $\mathbb{D}$, and $\mathbb{D}_A$, respectively, inheriting the properties $B' \cap \overline{S_A} = S_A$ and $\Phi(A' \oplus R, B') \subseteq \mathbb{Z}$, $\Phi(af, b) \notin \mathbb{Z}$.

Similarly, either

(1) $f \restriction R = k_R + \sum_i \Phi(-, b_i)x_i$ for some $k_R \in \mathbb{Z}$, $b_i \in B$, $x_i \in A \oplus R$; or

(2) there are pure extensions $B' = \langle B, b\rangle_*$ and $R' = \langle R, r\rangle_*$ in $\mathbb{D}$ and $\mathbb{D}_R$, respectively, inheriting the properties $B' \cap \overline{S_A} = S_A$ and $\Phi(A \oplus R', B') \subseteq \mathbb{Z}$, $\Phi(rf, b) \notin \mathbb{Z}$.

Proof. The proofs of the two statements are completely analogous, so we prove only the first one. Let us suppose that the second case does not hold, so $\Phi(af, b) \in \mathbb{Z}$ for all a and b.
Let $a \in \mathbb{D}_A(B)$. Applying Lemma 5.7 to $\Phi(af, -)$, B and $A' \oplus R$ with $A' = \langle A, a\rangle_*$, there are two possibilities. The first is an extension $B' = \langle B, b\rangle_*$ with $\Phi(af, b) \notin \mathbb{Z}$ but $\Phi(A' \oplus R, B') \subseteq \mathbb{Z}$ and $B' \cap \overline{S_A} = S_A$. By our assumption, this is not the case. Hence the second possibility must occur, which is $af \in A' \oplus R$.
Therefore $af \in \langle A, a\rangle_* \oplus R$ for all $a \in \mathbb{D}_A(B)$. Thus (ignoring the R-component) f induces an endomorphism of the $\mathbb{Q}$-vector space $\mathbb{D}_A(B)/A$, which is scalar multiplication on every element. So it is a multiplication by some rational number k/q with $q > 0$ relative prime to k. Hence $qf - k\colon \mathbb{D}_A(B) \to A \oplus R$, which by Lemma 5.6 must be of the form $qf - k = \sum_j \Phi(-, b_j)x_j$. There is a $\varphi\colon \mathbb{D}_A \to \mathbb{Z}$ and an $e \in \mathbb{D}_A(B)$ with $e\varphi = 1$ and $x_j\varphi = 0$ for all the finitely many j by Definition 5.1. So $q(ef)\varphi = k$, and hence q divides k, i.e., $q = 1$. Therefore f is of the claimed form on $\mathbb{D}_A(B)$, in particular $f \restriction A = k + \sum_j \Phi(-, b_j)x_j$. □

Finally, the version of the above lemma for killing a homomorphism by just knowing its name.

Lemma 5.10. *Let $\Phi\colon S \times S \to \mathbb{Z}$ be a bilinear map, such that the image of the left argument S is pure in S^* and the image of the right argument S is pure under the induced maps $S \to S_A^*$ and $S \to S_R^*$. In the ground model, let B, A, and R be pure subgroups of size less than $2^{\aleph_0}$ of $\mathbb{D}$, $\mathbb{D}_A$, and $\mathbb{D}_R$, containing S, S_A, and S_R, respectively. Furthermore, assume $B \cap \overline{S_A} = S_A$. Let f be a name of a homomorphism $S \to \mathbb{D}$. Then* in the ground model *there are countably generated extensions A', B', R' of A, B, R, which are still pure subgroups of $\mathbb{D}$, $\mathbb{D}_A$, and $\mathbb{D}_R$, respectively. Moreover, $B' \cap \overline{S_A} = S_A$ and it is forced that either*

(1) $f \restriction A' = k_A + \sum_j \Phi(-, b_{A,j})x_{A,j}$ and $f \restriction R' = k_R + \sum_i \Phi(-, b_{R,i})x_{R,i}$ for some $k_A, k_R \in \mathbb{Z}$, $b_{A,j}, b_{R,i} \in B'$, $x_{A,j}, x_{R,i} \in A' \oplus R'$; or

(2) $\Phi((A' \oplus R')f, B') \nsubseteq \mathbb{Z}$.

Proof. This follows from Lemma 5.9 the same way, as Lemma 5.8 follows from Lemma 5.7. So one chooses a maximal family of triples $(p_{A,i}, a_i, b_{A,i})$ and $(p_{R,j}, r_j, b_{R,j})$ such that the $p_{A,i}$ and $p_{R,j}$ are pairwise incompatible and the groups $A' \coloneqq \langle A, a_i : i\rangle_*$, $R' \coloneqq \langle R, r_j : j\rangle_*$, and $B' \coloneqq \langle B, b_{A,i}, b_{R,j} : i, j\rangle_*$ inherit the properties $\Phi(A' \oplus R', B') \subseteq \mathbb{Z}$ and $B' \cap \overline{S_A} = S_A$. Then an argument similar to Lemma 5.8 shows that these groups satisfy the claim of the present lemma. The proof is hence omitted. □

Finally, we have all the preparations to prove Proposition 5.2 at least for $\kappa = \aleph_1$.

PROOF OF PROPOSITION 5.2 FOR FORCING WITH $\mathrm{Fin}(\omega_1, 2)$. We construct A, R and $B = A^*$ simultaneously, as a union of continuous increasing chains A_α, R_α and B_α of pure subgroups of $\mathbb{D}$ of cardinality less than $2^{\aleph_0}$ for $\alpha < 2^{\aleph_0}$. Recall that the chain A_α is increasing if $A_\alpha \subseteq A_{\alpha+1}$ for all α, and the chain is continuous if $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for all limit α. Actually, we will have $S_A \subseteq A_\alpha \subseteq_* \mathbb{D}_A$, $S_R \subseteq R_\alpha \subseteq_* \mathbb{D}_R$, $S \subseteq B_\alpha \subseteq_* \mathbb{D}$, and also $B_\alpha \cap \overline{S_A} = S_A$. The duality will be given by a bilinear map $\Phi\colon S \times S \to \mathbb{Z}$ with the image of the left argument S pure in S^*, and the image of the right argument S pure in S_A^* and S_R^*. Moreover, $\Phi(S_R, S_A) = 0$.
For example, using Definition 5.1, we can define a dense Φ, which is still dense restricted to $S_A \times S$, $S_R \times S_R$ via

$$S_A \coloneqq \bigoplus_{n=0}^{\infty} \mathbb{Z} a_i \tag{5.10}$$

$$S_R \coloneqq \bigoplus_{n=0}^{\infty} \mathbb{Z} r_i \tag{5.11}$$

$$\Phi(r_i, r_j) \coloneqq \delta_{i,j} \tag{5.12}$$

$$\Phi(a_i, a_j) \coloneqq \delta_{i,j} \tag{5.13}$$

$$\Phi(r_i, a_j) \coloneqq 0 \tag{5.14}$$

$$\Phi(a_i, r_j) \coloneqq \begin{cases} 1 & \text{if } i - j = \pm 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{5.15}$$

We start with countable rank free groups:

$$A_0 \coloneqq S_A, \tag{5.16}$$

$$R_0 \coloneqq S_R, \tag{5.17}$$

$$B_0 \coloneqq S = S_A \oplus S_R. \tag{5.18}$$

Let $f_\alpha\colon \alpha < 2^{\aleph_0}$ be an enumeration of the names of all homomorphisms $S \to \mathbb{D}$ and $S \to \mathbb{Z}$ for the forcing poset $\mathrm{Fin}(\omega, 2)$. At every step α, we construct $A_{\alpha+1}$, $R_{\alpha+1}$ and $B_{\alpha+1}$ to "kill the homomorphism name f_α". This means that if f_α is a name of a homomorphism $S \to \mathbb{Z}$ then the unique extension of the homomorphism is either of the form $\Phi(x, -)$ for some $x \in A_{\alpha+1} \oplus R_{\alpha+1}$ or there is a $b \in B_{\alpha+1}$ with $bf_\alpha \notin \mathbb{Z}$. We can do this by Lemma 5.8. (In this case $A_{\alpha+1} = A_\alpha$ and $R_{\alpha+1} = R_\alpha$.)
Similarly, if f_α is a name of a homomorphism $S \to \mathbb{D}$ then the unique extension of the homomorphism is either a finite sum $\sum_i \Phi(-, b_i) y_i$ on A and $\sum_j \Phi(-, b'_j) y'_j$ on A for some $y_i, y'_j \in A_{\alpha+1} \oplus R_{\alpha+1}$ and $b_i, b'_j \in B_{\alpha+1}$ or there is an $x \in A_{\alpha+1} \oplus R_{\alpha+1}$ and $b \in B_{\alpha+1}$ with $\Phi(xf_\alpha, b) \notin \mathbb{Z}$. (ensuring that f_α will map $A \oplus R$ outside itself). This is possible by Lemma 5.10.
Thus, in the end, we will have $\operatorname{End} A \oplus R = \mathbb{Z}1_A \oplus \mathbb{Z}1_R \oplus (B \oplus B/S_R) \otimes (A \oplus R)$ and $B^* = A \oplus R$ via Φ. □

5.5. Adding arbitrary many Cohen real numbers. In this subsection, we extend the proof of Proposition 5.2 to all κ. Luckily, we don't have to change the construction for $\kappa = \aleph_1$.

PROOF OF PROPOSITION 5.2. The groups A, $B = A^*$, R constructed in the proof for the $\kappa = \aleph_1$ case work also for the extension forced with $\mathrm{Fin}(\kappa, 2)$. The reason is that for every group T in the ground model, every homomorphism $S \to T$ lies in a submodel, which is a forcing extension of the ground model forcing with $\mathrm{Fin}(\omega, 2)$. To see this, note that every homomorphism $S \to T$ is determined on the basis, so comes from a map $f : \omega \to T$. Every such map has a name using only countably many conditions from $\mathrm{Fin}(\kappa, 2)$, as the poset satisfies the countable chain condition. Let X be a countable subset of κ covering the domains of the conditions. Thus the name is also a name for the submodel given by the subposet $\mathrm{Fin}(X, 2)$, which is isomorphic to $\mathrm{Fin}(\omega, 2)$. □

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