2. Elements of linear programming

Contents

Consider two following examples.

2.1. Basic examples

Example 2.1. To find the maximal (minimal) value of the function (objective function)

$$z = x_1 + 3x_2$$

if the following constraints (restrictions) hold:

$$\begin{cases} -x_1 + x_2 \leqslant 1 \\ x_1 + 2x_2 \leqslant 8 \\ x_1 \geqslant 0, \ x_2 \geqslant 0 \ . \end{cases}$$

Usually this problem is formulated as follows:

$$z = x_1 + 3x_2 \to \max \quad (\to \min) \ ;$$

$$\begin{cases} -x_1 + x_2 \leqslant 1 \\ x_1 + 2x_2 \leqslant 8 \\ x_1 \geqslant 0, \ x_2 \geqslant 0 \ . \end{cases} \tag{2.1}$$

Consider the following graphical interpretation of the problem. On Picture 1 one can see the convex set D – the convex hull of 4 points with coordinates $(0;0)$, $(0;1)$, $(2;3)$ and $(8;0)$ (the so called *extremal* points of D). The set D is just the set of points whose coordinates satisfy the system of constraints (2.1). It is called the set of admissible solutions or polyhedron (polygon) of solutions.

Also one can see the line $l : x_1 + 3x_2 = c$ where c is an appropriate constant. Since we want to find the maximal value of c we should move this line in the direction of the vector $(1;3)$ (the arrow on Figure 1). In any case the line l must intersect the set D.

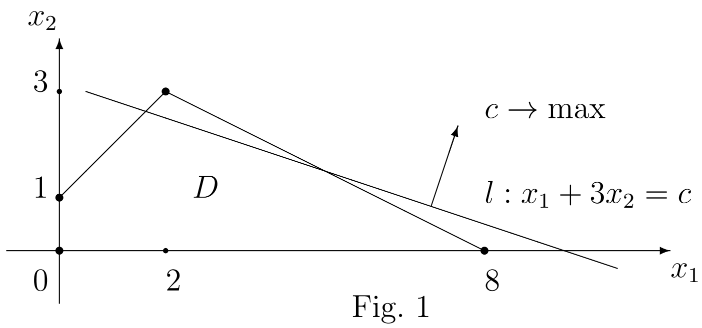

Fig. 1

On Figure 2 one can see the line $l : x_1 + 3x_2 = 11$ and the value $c = 11$ is the maximal value. For a bigger value of c the line l will not intersect the set D.

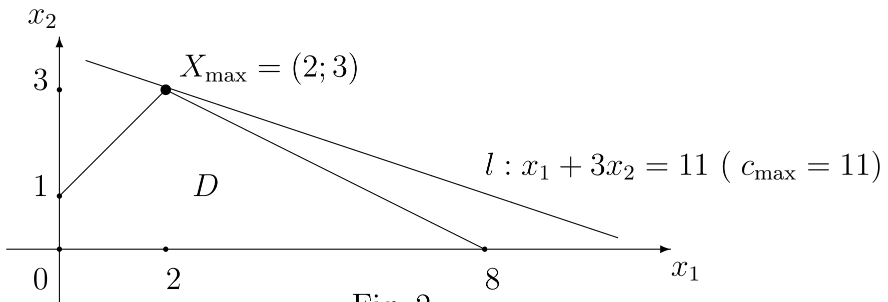

Fig. 2

Thus, the optimal solution is $X_{\max} = (2; 3)$. This is one of 4 extremal points of the set D.

On Figure 3 one can see the line $l : x_1 + 3x_2 = 0$. The value $c = 0$ is the minimal value. For a smaller value of c the line l will not intersect the set D. The optimal solution is $X_{\min} = (0; 0)$. The corresponding point is again the extremal point of D.

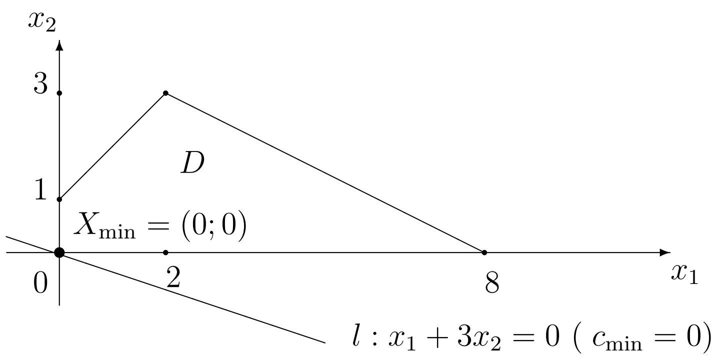

Fig. 3

Example 2.2. An enterprise E deals with three types of resources, say, a material, equipment and electric energy. E and can organize a production in two different ways. The expenditure of resources and the amortization of equipment, the total resource expenditure (*per month*) for each technology of producing is given in the following table:

resource	expenditure per month		total
	technology 1	technology 2	amount
material	1	2	4
equipment	1	1	3
energy	2	1	8

The enterprise produces 3 thousand items of production for the first technology and 4 thousand items for the second one. During how many months should the enterprise work in each way in order to reach the maximal output of production?

Solution. Let us describe a mathematical model of the problem. For this purpose we introduce the following notation:

- x_1 stands for the duration (*in months*) of producing for the technology 1;
- x_2 stands for the duration of producing for the technology 2.

Now we can formulate the problem of linear optimization as follows

$$z = 3x_1 + 4x_2 \to \max \quad ;$$

$$\begin{cases} x_1 + 2x_2 \leqslant 4 \ , \\ x_1 + x_2 \leqslant 3 \ , \\ 2x_1 + x_2 \leqslant 8 \ ; \\ x_1 \geqslant 0, \ x_2 \geqslant 0 \ . \end{cases}$$

The first three inequalities of the system mean that the total amount of each kind of resources cannot be exceeded. The solution of this problem given below in the subsection 2.4 is

$$x_1 = 2 \ , \ x_2 = 1 \ .$$

It follows that for 2 months the enterprise should work using the first technology and 1 month using the second one. The maximal number of produced items (maximal output) is

$$3 \cdot 2 + 4 \cdot 1 = 10$$

that is, 10 thousand items.

2.2. The linear programming problem

The general linear programming (LP) problem can be formulated as follows:

$$z = \sum_{j=1}^{n} c_j \, x_j \to \min \ \ (\to \max)$$

$$S : \begin{cases} \sum\limits_{j=1}^{n} a_{1j} \, x_j \leqslant b_1 \\ \sum\limits_{j=1}^{n} a_{2j} \, x_j \leqslant b_2 \\ \\ \dots \\ \sum\limits_{j=1}^{n} a_{mj} \, x_j \leqslant b_m \ . \end{cases} \tag{2.2}$$

Here the system S of constraints consists of linear inequalities of the type $\sum\limits_{j=1}^{n} a_{kj} \, x_j \leqslant b_k$. Note that

1. $\sum\limits_{j=1}^{n} a_{kj} \, x_j \geqslant b_k \quad \Leftrightarrow \quad \sum\limits_{j=1}^{n} -a_{kj} \, x_j \leqslant -b_k \, .$

2. $\sum\limits_{j=1}^{n} a_{kj} \, x_j = b_k \quad \Leftrightarrow \quad \begin{cases} \sum\limits_{j=1}^{n} a_{kj} \, x_j \leqslant b_k \\ \sum\limits_{j=1}^{n} -a_{kj} \, x_j \leqslant -b_k \ . \end{cases}$

Thus, any system of linear constraints (equalities or inequalities) can be written in the form (2.2).

The canonical form of the linear programming problem is as follows

$$z = \sum_{j=1}^{n} c_j\, x_j \to \min \ \ (\to \max)$$

$$S_{canon}: \begin{cases} \sum\limits_{j=1}^{n} a_{1j}\, x_j = b_1 \\ \sum\limits_{j=1}^{n} a_{2j}\, x_j = b_2 \\ \\ \dots \\ \sum\limits_{j=1}^{n} a_{mj}\, x_j = b_m\ . \\ x_1 \geqslant 0,\ x_2 \geqslant 0\ , \dots,\ x_n \geqslant 0\ . \end{cases} \tag{2.3}$$

Reducing the LP problem to a canonical form.

Consider the following problem.

$$z = 2x_1 - x_2 + 5x_3 \to \min$$

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 \geqslant -1 \\ -x_1 + x_2 + 2x_3 \leqslant \ 2 \\ x_1 + 2x_2 - 3x_3 = \ \ 1 \end{cases}$$

$$x_1 \geqslant 0\ ,\ x_2 \geqslant 0\ ,\ x_3 \geqslant 0\ .$$

If we need to rewrite this problem in a canonical form we do the following transformations. For the first inequality we introduce the additional nonnegative variable x_4, for the second inequality the nonnegative variable x_5 and write

$$z = 2x_1 - x_2 + 5x_3 \to \min$$

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 - x_4 = -1 \\ -x_1 + \ x_2 + 2x_3 \quad + x_5 = \ \ 2 \\ x_1 + 2x_2 - 3x_3 = \ \ 1 \end{cases}$$

$$x_1 \geqslant 0\ ,\ \dots\ ,\ x_5 \geqslant 0\ .$$

Pay attention to the signs by x_4 and x_5.

In the first equation $b_1 = -1 < 0$. Then we multiply it by -1 and obtain the canonical form

$$z = 2x_1 - x_2 + 5x_3 \to \min$$

$$\begin{cases} -3x_1 + 2x_2 + 5x_3 + x_4 = 1 \\ -x_1 + \ x_2 + 2x_3 \quad + x_5 = 2 \\ x_1 + 2x_2 - 3x_3 = 1 \end{cases}$$

$$x_1 \geqslant 0\ ,\ \dots\ ,\ x_5 \geqslant 0\ .$$

For instance, let us rewrite the problem (2.1) of the basic example

$$z = x_1 + 3x_2 \to \max\ ;$$

$$\begin{cases} -x_1 + x_2 \leqslant 1 \\ x_1 + 2x_2 \leqslant 8 \\ x_1 \geqslant 0,\ x_2 \geqslant 0\ . \end{cases}$$

in a canonical form. For this purpose two additional nonnegative variables x_3 and x_4 are introduced (one for each inequality) :

$$z = x_1 + 3x_2 \to \max\ ;$$

$$\begin{cases} -x_1 + x_2 + x_3 \qquad = 1 \\ x_1 + 2x_2 \qquad + x_4 = 8 \\ x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 \geqslant 0\ . \end{cases}$$

The canonical form is more convenient for the algorithm known as *simplex algorithm* in order to solve the problem. There is a module "simplex" in the MAPLE package and if you want to solve either general or canonical problem first of all you should include this module for computations.

2.3. Short geometric description of the simplex algorithm

Let us return to the canonical form of the linear programming problem (2.3). It is not hard to see that the set D of admissible solutions for the system S_{canon} is a convex closed subset of the space $\mathbb{R}^n$ (it is also true for the set of solutions D of the system S in (2.2) – see Figure 1). It is often called a polyhedron (a polygon, if $n = 2$) of solutions for the LP problem.

Note the set D could be empty set or be unbounded, for instance, D could be a half-space.

We suppose that D is not empty. If D is a bounded set then D is a *convex hull* of its extremal points. Recall that the point $A \in D$, $D \in \mathbb{R}^n$, is called *extremal* if A is not an inner point of any line segment BC with the end points B and C belonging to D. For instance, the set D on Picture 1 is the convex hull of four extremal points with coordinates $(0; 0)$, $(0; 1)$, $(2; 3)$ and $(8; 0)$.

It can be proved that the optimal solution of the canonical problem corresponds to one of the extremal points of D. Note the there may be several extremal points with the same value of the function z, that is, the problem of linear optimization may have not a unique solution (however, the numbers $z_{\min}$ and $z_{\max}$ are unique). Also note that $z_{\min}$ may be equal to $-\infty$ or $z_{\max}$ may be equal to $+\infty$ if D is an unbounded set.

A short geometric description of the simplex algorithm.

1. One finds a basic or initial solution (sometimes it is called a basic plan). Geometrically it corresponds to an extremal point of the set D if D is non-empty. In general, this basic solution is not an optimal one.
2. Starting from the basic solution one comes step by step to the optimal solution (minimal or maximal one). Each step is a transition from one extremal point to the adjacent extremal point of the polyhedron of solutions. For this adjacent extremal point the objective function z increases (for maximum) or decreases (for minimum) as much as possible.

Algebraically, the information about the current extremal point for each step is contained in so called *simplex tableau* (see below Subsection 2.5). For each step of simplex algorithm one needs to make a pivot transform of the simplex tableau choosing a so called *leading* or *pivot* element.

2.4. Linear programming. Computations in MAPLE

Solution of the LP problem for Example 2.1.

```
> with(simplex):
```

```
Warning, the protected names maximize and minimize have been
redefined and unprotected
```

```
> x:=array(1..2);
```

$$x := \mathrm{array}(1..2, [])$$

```
> z:=(x)->x[1]+3*x[2];
```

$$z := x \to x_1 + 3\,x_2$$

```
> sys:={-x[1]+x[2]<=1,x[1]+2*x[2]<=8};
```

$$sys := \{x_1 + 2\,x_2 \leq 8,\ -x_1 + x_2 \leq 1\}$$

```
> Xmax:=maximize(z(x),sys,NONNEGATIVE);
```

$$Xmax := \{x_2 = 3,\ x_1 = 2\}$$

```
> Zmax:=z([2,3]);
```

$$Zmax := 11$$

```
> Xmin:=minimize(z(x),sys,NONNEGATIVE);
```

$$Xmin := \{x_1 = 0,\ x_2 = 0\}$$

```
> Zmin:=z([0,0]);
```

$$Zmin := 0$$

Solution of the problem in Example 2.2.

```
> with(simplex):
```

```
Warning, the protected names maximize and minimize have been
redefined and unprotected
```

```
> x:=array(1..2);
```

$$x := \mathrm{array}(1..2, [])$$

```
> z:=(x)->3*x[1]+4*x[2];
```

$$z := x \to 3\,x_1 + 4\,x_2$$

```
> sys:={x[1]+2*x[2]<=4,x[1]+x[2]<=3,2*x[1]+x[2]<=8};
```

$$sys := \{x_1 + 2\,x_2 \leq 4,\ x_1 + x_2 \leq 3,\ 2\,x_1 + x_2 \leq 8\}$$

```
> maximize(z(x),sys,NONNEGATIVE);
```

$$\{x_1 = 2,\ x_2 = 1\}$$

```
> z([2,1]);
```

$$10$$

2.5. Simplex algorithm

Consider the linear programming problem from Example 2.2.

$$z = 3x_1 + 4x_2 \to \max \quad ;$$

$$\begin{cases} x_1 + 2x_2 \leqslant 4 \ , \\ x_1 + x_2 \leqslant 3 \ , \\ 2x_1 + x_2 \leqslant 8 \ ; \\ x_1 \geqslant 0, \ x_2 \geqslant 0 \ . \end{cases}$$

First of all, we reduce it to the canonical form introducing 3 additonal variables x_3, x_4 and x_5 (one for each constraint). We get

$$z = 3x_1 + 4x_2 \to \max \quad ;$$

$$\begin{cases} x_1 + 2x_2 & + & x_3 & & & & & = & 4 \ , \\ x_1 + x_2 & & & + & x_4 & & & = & 3 \ , \\ 2x_1 + x_2 & & & & & + & x_5 & = & 8 \ ; \end{cases} \tag{2.4}$$

$$x_j \geqslant 0, \quad j = 1 \dots 5 \ .$$

Write down the following initial *simplex tableau.*

	c_j	3	4	0	0	0	
c	basis	x_1	x_2	x_3	x_4	x_5	b
$c_3 = 0$	x_3	1	2	1	0	0	4
$c_4 = 0$	x_4	1	1	0	1	0	3
$c_5 = 0$	x_5	2	1	0	0	1	8
	Δ_j	-3	-4	0	0	0	$z = 0$

Here the entry c_j is the coefficient by x_j in the objective function, h_{ij} is the coefficient by x_j in the i-th constraint of (2.4) (for instance, $h_{11} = 1$, $h_{12} = 2$, $h_{13} = 1$, $\dots$. The matrix (h_{ij}) is situated in the middle of the simplex tableau). The coefficients b_i are in the right-hand side of the constraints.

New entries obtained after a current iteration of the simplex algorithm are still denoted h_{ij} (sometimes, h_{ij}^{new}). The coefficients in basic columns form the identity submatrix (in the above tableau the basic columns correspond to x_3, x_4 and x_5, the word "basis" means "basis of columns"). Also we have the initial estimates

$$\Delta_j = \sum_{i=1}^{3} h_{ij} c_i - c_j \ , \qquad j = 1 \dots 5 \ ,$$

and the evaluation

$$z(X) = \sum_{i=1}^{3} c_i b_i \ .$$

Now we have a basic solution (plan)

$$X_0 = (0, 0, 4, 3, 8) \ , \qquad z(X_0) = 0 \ .$$

It is determined from the simplex tableau choosing columns

basis	b
x_3	4
x_4	3
x_5	8

The other variables (that is, x_1 and x_2) have trivial values. Note that b_j must be nonnegative.

However, this solution is not the optimal one since in the last row we have two negative numbers: $\Delta_1 = -3$, $\Delta_2 = -4$. The condition

$$\Delta_j \geqslant 0 \quad \forall j$$

must be satisfied for the optimal solution (if $z \to \max$).

Step 1 of the algorithm

Every step of the simplex algorithm consists in transforming the simpex tableau. For this purpose one should determine a so called *leading* or *resolving element*. The usual way to find the leading element is as follows. Let us calculate ratios

$$\theta_{ij} = b_i/h_{ij} \qquad (b_i \geqslant 0)$$

for all columns with negative Δ_j. Note that if the entry $h_{ij} \leqslant 0$ then the ratio θ_{ij} is omitted. In our case

$\theta_{11} = 4$	$\theta_{12} = 2$
$\theta_{21} = \mathbf{3}$	$\theta_{22} = 3$
$\theta_{31} = 4$	$\theta_{32} = 8$
$\Delta_1 < 0$	$\Delta_2 < 0$

We find the minimal ratio $\mu_j = \min\limits_i \theta_{ij}$ for the j-th column. We have

$$\mu_1 = \min_i \theta_{i1} = \theta_{21} = 3\,, \;\; \mu_2 = \min_i \theta_{i2} = \theta_{12} = 2\,.$$

Then we choose the number j_0 such that the value

$$-\mu_{j_0}\Delta_{j_0}$$

is maximal for all the numbers $-\mu_j\Delta_j$ provided $\Delta_j < 0$.

We have

$$-\mu_1\Delta_1 = 9\,, \quad -\mu_2\Delta_2 = 8\,.$$

Consequently, $j_0 = 1$. The column number j_0 will be called the leading column. If there were two or more leading columns then one might choose an arbitrary one.

We saw for the leading column that $\mu_1 = \min\limits_i \theta_{i1} = \theta_{21} = 3$. Set $i_0 = 2$. This is the number of row with minimal value of θ_{i,j_0} in the leading column. The row number i_0 is called the leading row.

By definition, the leading element is the entry h_{i_0,j_0}, that is, $h_{21} = 1$ in our case. The leading (pivot) element typed in boldface below is standing on the intersection of leading row (row labelled x_4) and the leading column (column labelled x_1).

	c_j	3	4	0	0	0	
c	basis	x_1	x_2	x_3	x_4	x_5	b
$c_3 = 0$	x_3	1	2	1	0	0	4
$c_4 = 0$	x_4	**1**	1	0	1	0	3
$c_5 = 0$	x_5	2	1	0	0	1	8
	Δ_j	-3	-4	0	0	0	$z = 0$

Then we make the pivot transform (see Remark 3 below) using the elementary transformations of rows and obtain the second simplex table

	c_j	3	4	0	0	0	
c	basis	x_1	x_2	x_3	x_4	x_5	b
$c_3 = 0$	x_3	0	1	1	-1	0	1
$c_1 = 3$	x_1	1	1	0	1	0	3
$c_5 = 0$	x_5	0	-1	0	-2	1	2
	Δ_j	0	-1	0	3	0	$z = 9$

Note that we substitute x_1 for x_4 in the "basis" column since the leading row corresponds to x_4 and leading column corresponds to x_1 in the previous tableau. In the current simplex tableau the columns labelled x_3, x_1, x_5 form the identity submatrix.

Thus, we have obtained the next solution

$$X_1 = (3, 0, 1, 0, 2) \ , \qquad z(X_1) = 9 \ ,$$

but it is not still an optimal one: $\Delta_2 = -1 < 0$.

Step 2 of the algorithm

We determine the leading element at the second step.

$\theta_{12} = \mathbf{1}$
$\theta_{22} = 3$
θ_{32} : $-$(omitted)
$\Delta_2 < 0$

We have

$$\mu_2 = \min_i \theta_{ij} = \theta_{12} = 1 \ .$$

The leading element $h_{12} = 1$ stands on the intersection of the leading row (row labelled x_3) and the leading column (column labelled x_2).

	c_j	3	4	0	0	0	
c	basis	x_1	x_2	x_3	x_4	x_5	b
$c_3 = 0$	x_3	0	**1**	1	-1	0	1
$c_1 = 3$	x_1	1	1	0	1	0	3
$c_5 = 0$	x_5	0	-1	0	-2	1	2
	Δ_j	0	-1	0	3	0	$z = 9$

The pivot transform yields

	c_j	3	4	0	0	0	
c	basis	x_1	x_2	x_3	x_4	x_5	b
$c_2 = 4$	x_2	0	1	1	-1	0	1
$c_1 = 3$	x_1	1	0	-1	2	0	2
$c_5 = 0$	x_5	0	0	1	-3	1	3
	Δ_j	0	0	1	2	0	$z = 10$

Here all $\Delta_i \geqslant 0$. It follows that we have determined the optimal solution

$$X_{\text{opt}} = (2, 1, 0, 0, 3) \ , \qquad z(X_{\text{opt}}) = 10 \ .$$

Discarding the additional variables we get

$$x_1 = 2,\ x_2 = 1\ , \quad z_{\max} = 10\ ,$$

that is the solution of the original problem 2.

Remark 1. If in the linear programming problem we minimize the objective function, $z \to \min$, then the conditions $\Delta_j \leqslant 0$ for all j must be satisfied for the optimal solution.

If the conditions $\Delta_j \leqslant 0$ are not satisfied for all j then the current solution is not optimal and we choose the number j_0 of the leading column in such a way that the value

$$\mu_{j_0}\Delta_{j_0}$$

is maximal for all the values $\mu_j\Delta_j$ provided $\Delta_j > 0$. The leading row is determined as above for the case $z \to \max$ as well as the pivot transform of the simplex tableau.

Remark 2. If in a simplex tableau $\Delta_j < 0$ for some j and $h_{ij} \leqslant 0$ for all i then $z_{\max} = +\infty$.

Remark 3. The pivot transform can be done as follows (by means of so called *rectangle rule*). Let $h_{i_0 j_0}$ be the leading element. If the element h_{ij} doesn't belong to the leading row or leading column (that is, $i \neq i_0$ and $j \neq j_0$) and b_i doesn't belong to the leading row ($i \neq i_0$) then

$$h_{ij}^{new} = h_{ij} - \frac{h_{i_0 j} \cdot h_{i j_0}}{h_{i_0 j_0}}\ , \quad b_i^{new} = b_i - \frac{b_{i_0} \cdot h_{i j_0}}{h_{i_0 j_0}}\ .$$

If $i = i_0$ then

$$h_{i_0 j}^{new} = \frac{h_{i_0 j}}{h_{i_0 j_0}}\ , \quad b_{i_0}^{new} = \frac{b_{i_0}}{h_{i_0 j_0}}\ .$$

If $i \neq i_0$ and $j = j_0$ then

$$h_{i j_0}^{new} = 0\ .$$

The new elements Δ_j^{new} and z^{new} can be obtained in the similar way

$$\Delta_j^{new} = \Delta_j - \frac{\Delta_{j_0}}{h_{i_0 j_0}}\, h_{i_0 j}\ , \qquad z^{new} = z - \frac{\Delta_{j_0}}{h_{i_0 j_0}}\, b_{i_0}\ .$$

In other words, the leading row H_{i_0} (including b_{i_0}) is divided by the pivot element $h_{i_0 j_0} > 0$,

$$H_{i_0}^{new} = \frac{1}{h_{i_0 j_0}}\, H_{i_0}\ , \quad b_{i_0}^{new} = \frac{b_{i_0}}{h_{i_0 j_0}}\ .$$

If $i \neq i_0$ then the new i-th row

$$H_i^{new} = H_i - \frac{h_{i j_0}}{h_{i_0 j_0}}\, H_{i_0}\ , \quad b_i^{new} = b_i - \frac{h_{i j_0}}{h_{i_0 j_0}}\, b_{i_0}\ .$$

The last row (Δ, z) of the simplex tableau is transformed in a similar way,

$$\Delta^{new} = \Delta - \frac{\Delta_{j_0}}{h_{i_0 j_0}}\, H_{i_0}\ , \qquad z^{new} = z - \frac{\Delta_{j_0}}{h_{i_0 j_0}}\, b_{i_0}\ .$$

Remark 4. In general, the initial simplex tableau for the problem (2.3) or the equivalent problem

$$z = \sum_{j=1}^{n} c_j\, x_j \to \max$$

$$\begin{cases} \sum\limits_{j=1}^{n} a_{1j}\, x_j + x_{n+1} = b_1 \\ \sum\limits_{j=1}^{n} a_{2j}\, x_j + x_{n+2} = b_2 \\ \\ \dots \\ \sum\limits_{j=1}^{n} a_{mj}\, x_j + x_{n+m} = b_m \; ; \\ x_1 \geqslant 0, \; , \dots , \; x_n \geqslant 0 \; , \dots , \; x_{n+m} \geqslant 0. \end{cases} \tag{2.5}$$

(it is assumed that $b_i \geqslant 0$ in (2.5)) is as follows

	c_j	c_1	$\dots$	c_n	0	$\dots$	0	
c	basis	x_1	$\dots$	x_n	x_{n+1}	$\dots$	x_{n+m}	b
$c_{n+1} = 0$	x_{n+1}	$h_{11} = a_{11}$	$\dots$	$h_{1n} = a_{1n}$	1	$\dots$	0	b_1
$\dots$	$\dots$	$\dots$		$\dots$	0		0	$\dots$
$c_{n+m} = 0$	x_{n+m}	$h_{m1} = a_{m1}$	$\dots$	$h_{mn} = a_{mn}$	0	$\dots$	1	b_m
	Δ_j	Δ_1	$\dots$	Δ_n	0	$\dots$	0	$z = 0$

As a basic solution the array

$$X_0 = (0, \dots , 0, b_1, \dots\, b_m)$$

is taken.

Further, the simplex algorithm works as it was described above until the conditions $\Delta_j \geqslant 0$, $j = 1, \dots , n + m$, are satisfied (and we obtain an optimal solution, $z = z_{\max}$) or we get the situation $z_{\max} = +\infty$ (see Remark 2).

2.6. More examples

Example 2.3. Consider the following LP problem:

$$z = 2x_1 + x_2 - 5x_3 \to \text{extr}$$

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 \geqslant -1 \\ -x_1 + x_2 + 2x_3 \leqslant 2 \\ x_1 + 2x_2 - 3x_3 = 1. \end{cases}$$

Note that the constraints are of mixed type (one equality and two inequalities) and that there are **no** constraints $x_i \geqslant 0$.

Computations in MAPLE.

```
> with(simplex):

Warning, the protected names maximize and minimize have been
redefined and unprotected
```

```
> x:=array(1..3);
```

$$x := \text{array}(1..3, [])$$

```
> z:=(x)->2*x[1]+x[2]-5*x[3];
```

$$z := x \to 2\,x_1 + x_2 - 5\,x_3$$

```
> sys:=
> {3*x[1]-2*x[2]-5*x[3]>=-1,-x[1]+x[2]+2*x[3]<=2,x[1]+2*x[2]-3*x[3]=1
> };
```

$$sys := \{-1 \leq 3\,x_1 - 2\,x_2 - 5\,x_3,\ -x_1 + x_2 + 2\,x_3 \leq 2,\ x_1 + 2\,x_2 - 3\,x_3 = 1\}$$

```
> X1:=minimize(z(x),sys);
```

$$X1 := \{x_2 = 2,\ x_3 = 3,\ x_1 = 6\}$$

```
> Zmin:=z([6,2,3]);
```

$$Zmin := -1$$

```
> X2:=maximize(z(x),sys);
```

$$X2 :=$$

Hence, $z_{\min} = z(6, 2, 3) = -1$, $z_{\max} = +\infty$.

Example 2.4. This example shows the difference between the LP problem *with* the constraints $x_i \geqslant 0$ and *without* them.

Consider two LP problems:

Case 1.

$$z = 2x_1 - x_2 + 5x_3 \to \min$$

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 \geqslant -1 \\ -x_1 + x_2 + 2x_3 \leqslant 2 \\ x_1 + 2x_2 - 3x_3 = 1. \end{cases}$$

Case 2.

$$z = 2x_1 - x_2 + 5x_3 \to \min$$

$$\begin{cases} 3x_1 - 2x_2 - 5x_3 \geqslant -1 \\ -x_1 + x_2 + 2x_3 \leqslant 2 \\ x_1 + 2x_2 - 3x_3 = 1\ , \\ x_1 \geqslant 0\ ,\ x_2 \geqslant 0\ ,\ x_3 \geqslant 0\ . \end{cases}$$

In Case 1 we have $z_{\min} = -\infty$, in Case 2 we obtain $z_{\min} = z(0, 1/2, 0) = -1/2$

Computations in MAPLE.

```
> with(simplex):

Warning, the protected names maximize and minimize have been
redefined and unprotected

> x:=array(1..3);
```

$$x := \text{array}(1..3, [])$$

```
> z:=(x)->2*x[1]-x[2]+5*x[3];
```

$$z := x \to 2\,x_1 - x_2 + 5\,x_3$$

```
> sys:=
> {3*x[1]-2*x[2]-5*x[3]>=-1,-x[1]+x[2]+2*x[3]<=2,x[1]+2*x[2]-3*x[3]=1
> };
```

$$sys := \{-1 \leq 3\,x_1 - 2\,x_2 - 5\,x_3,\ -x_1 + x_2 + 2\,x_3 \leq 2,\ x_1 + 2\,x_2 - 3\,x_3 = 1\}$$

```
> X1:=minimize(z(x),sys);
```

$$X1 :=$$

```
> X2:=minimize(z(x),sys,NONNEGATIVE);
```

$$X2 := \{x_1 = 0,\ x_3 = 0,\ x_2 = \frac{1}{2}\}$$

```
> z([0,1/2,0]);
```

$$\frac{-1}{2}$$

Hence, $z_{\min} = -\infty$ in Case 1 meanwhile $z_{\min} = -1/2$ in Case 2.

Example 2.5. In this example the set of admissible solutions is empty.

$$z = 2x_1 - x_2 + 6x_3 \to \min$$

$$\begin{cases} x_1 + x_2 + 2\,x_3 \leqslant -3 \\ 5\,x_1 + 2\,x_2 + 4\,x_3 \leqslant 7\,, \\ x_1 \geqslant 0\ ,\ x_2 \geqslant 0\ ,\ x_3 \geqslant 0\ . \end{cases}$$

Computations in MAPLE.

```
> with(simplex):

Warning, the protected names maximize and minimize have been
redefined and unprotected

> x:=array(1..3);
```

$$x := \mathrm{array}(1..3,\ [])$$

```
> z:=(x)->2*x[1]-x[2]+6*x[3];
```

$$z := x \to 2\,x_1 - x_2 + 6\,x_3$$

```
> sys:={x[1]+x[2]+2*x[3]<=-3,5*x[1]+2*x[2]+4*x[3]<=7};
```

$$sys := \{x_1 + x_2 + 2\,x_3 \leq -3,\ 5\,x_1 + 2\,x_2 + 4\,x_3 \leq 7\}$$

```
> minimize(z(x),sys,NONNEGATIVE);
```

$$\{\ \}$$

The output { } means that the system of constraints is inconsistent.

2.7. The M-method or the method of artificial basis

Assume that we are given an LP problem

$$z = \sum_{j=1}^{n} c_j\, x_j \to \min$$

$$
S_{canon}: \begin{cases} \sum_{j=1}^{n} a_{1j}\, x_j = b_1 \\ \sum_{j=1}^{n} a_{2j}\, x_j = b_2 \\ \\ \dots \\ \sum_{j=1}^{n} a_{mj}\, x_j = b_m \; ; \\ x_1 \geqslant 0,\; x_2 \geqslant 0\,, \dots,\; x_n \geqslant 0\,, \end{cases} \tag{2.6}
$$

where $b_i \geqslant 0$, and there is no identity $m \times m$ submatrix in $A = (a_{ij})$. Thus, we need to determine the basic solution (plan).

The M-method or the method of artificial basis solves this problem.

Consider the *extended* LP problem

$$
z = \sum_{j=1}^{n} c_j\, x_j + M x_{n+1} + \dots + M x_{n+m} \to \min
$$

$$
S_{ext}: \begin{cases} \sum_{j=1}^{n} a_{1j}\, x_j + x_{n+1} = b_1 \\ \sum_{j=1}^{n} a_{2j}\, x_j + x_{n+2} = b_2 \\ \\ \dots \\ \sum_{j=1}^{n} a_{mj}\, x_j + x_{n+m} = b_m \; ; \\ x_1 \geqslant 0,\; x_2 \geqslant 0\,, \dots,\; x_{n+m} \geqslant 0\,. \end{cases} \tag{2.7}
$$

Here we introduced the *artificial* variables $x_{n+1}, \dots,\; x_{n+m}$ (one for each constraint). The parameter M is viewed as a sufficiently large positive number ($M \gg 1$).

For the corresponding maximization problem

$$
z = \sum_{j=1}^{n} c_j\, x_j \to \max
$$

$$
S_{canon}: \begin{cases} \sum_{j=1}^{n} a_{1j}\, x_j = b_1 \\ \sum_{j=1}^{n} a_{2j}\, x_j = b_2 \\ \\ \dots \\ \sum_{j=1}^{n} a_{mj}\, x_j = b_m \; ; \\ x_1 \geqslant 0,\; x_2 \geqslant 0\,, \dots,\; x_n \geqslant 0\,, \end{cases} \tag{2.8}
$$

the extended problem is stated as follows ($M \gg 1$):

$$
z = \sum_{j=1}^{n} c_j\, x_j - M x_{n+1} - \dots - M x_{n+m} \to \max
$$

$$
S_{ext}: \begin{cases} \sum\limits_{j=1}^{n} a_{1j}\, x_j + x_{n+1} = b_1 \\ \sum\limits_{j=1}^{n} a_{2j}\, x_j + x_{n+2} = b_2 \\ \\ \dots \\ \sum\limits_{j=1}^{n} a_{mj}\, x_j + x_{n+m} = b_m \; ; \\ x_1 \geqslant 0,\; x_2 \geqslant 0\,, \dots,\; x_{n+m} \geqslant 0\,. \end{cases} \tag{2.9}
$$

One can prove the following theorem.

Theorem. If in the optimal solution

$$
\widehat{X}_{\text{opt}} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}, 0, \dots, 0)
$$

of the extended LP problem (2.7) or (2.9) all artificial variables are trivial then the solution

$$
X_{\text{opt}} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})
$$

is the optimal solution for the original LP problem (2.6) or (2.8).

Remark. If in the optimal solution $\widehat{X}$ at least one artificial variable is non-trivial then the system of constraints of the original problem (2.6) or (2.8) is inconsistent.

Example 2.6. In order to illustrate the M-method let us consider the following problem.

$$
z = -x_1 - 2x_2 + 3x_3 - x_4 \to \min
$$

$$
\begin{cases} 2x_1 + \;\; x_2 - 2\,x_3 \;\; - x_4 \;\; = \;\; 5 \\ \;\; x_1 + 3\,x_2 \qquad\quad + 3x_4 \;\; = \;\; 2\,. \\ x_1 \geqslant 0\,,\; \dots\;,\; x_4 \geqslant 0\,. \end{cases}
$$

First, transform this LP problem to the extended problem:

$$
z = -x_1 - 2x_2 + 3x_3 - x_4 + Mx_5 + Mx_6 \to \min
$$

$$
\begin{cases} 2x_1 + \;\; x_2 - 2\,x_3 \;\; - x_4 + x_5 \qquad\quad = \;\; 5 \\ \;\; x_1 + 3\,x_2 \qquad\quad + 3x_4 \qquad + x_6 \;\; = \;\; 2 \\ x_1 \geqslant 0\,,\; \dots\;,\; x_6 \geqslant 0\,. \end{cases}
$$

with the artificial variables x_5, x_6.

The initial simplex tableau is composed as follows:

	c_j	-1	-2	3	-1	M	M	
c	basis	x_1	x_2	x_3	x_4	x_5	x_6	b
$c_5 = M$	x_5	2	1	-2	-1	1	0	5
$c_6 = M$	x_6	1	3	0	3	0	1	2
Δ_j	$\times 1$	1	2	-3	1	0	0	$z = 7M$
	$\times M$	3	4	-2	2	0	0	

The last two rows are for Δ_j which are of the linear form $\Delta_j = q_j + p_j M$. In the last row the j-th entry is equal to p_j and it should be multiplied by M. For instance, $\Delta_1 = 1 + 3 \cdot M$,

$\Delta_2 = 2 + 4 \cdot M$ and so on. The condition $\Delta_j \leqslant 0$ must be satisfied for the optimal solution (if $z \to \min$).

The iterations of the simplex algorithm will be fulfilled as described above. The leading element is $h_{12} = 1$.

	c_j	-1	-2	3	-1	M	M	
c	basis	x_1	x_2	x_3	x_4	x_5	x_6	b
$c_5 = M$	x_5	2	1	-2	-1	1	0	5
$c_6 = M$	x_6	**1**	3	0	3	0	1	2
Δ_j	$\times 1$	1	2	-3	1	0	0	$z = 7M$
	$\times M$	3	4	-2	2	0	0	

The transform yields

	c_j	-1	-2	3	-1	M	M	
c	basis	x_1	x_2	x_3	x_4	x_5	x_6	b
$c_5 = M$	x_5	0	-5	-2	-7	1	-2	1
$c_1 = -1$	x_1	1	3	0	3	0	1	2
Δ_j	$\times 1$	0	-1	-3	-2	0	-1	$z = M - 2$
	$\times M$	0	-5	-2	-7	0	-3	

Since all $\Delta_j \leqslant 0$ the solution

$$\widehat{X} = (2, 0, 0, 0, 1, 0)$$

of the extended problem is optimal. The original system of constraints is inconsistent because the artificial variable x_5 in the optimal solution is non-trivial.

Computations in MAPLE.

```
> with(simplex):

Warning, the protected names maximize and minimize have been
redefined and unprotected

> x:=array(1..4);
```

$$x := \text{array}(1..4, [])$$

```
> z:=(x)->-x[1]-2*x[2]+3*x[3]-x[4];
```

$$z := x \to -x_1 - 2\,x_2 + 3\,x_3 - x_4$$

```
> sys:={2*x[1]+x[2]-2*x[3]-x[4]=5,x[1]+3*x[2]+3*x[4]=2};
```

$$sys := \{2\,x_1 + x_2 - 2\,x_3 - x_4 = 5,\ x_1 + 3\,x_2 + 3\,x_4 = 2\}$$

```
> minimize(z(x),sys,NONNEGATIVE);
```

$$\{\,\}$$