# 3. Ordinary differential equations

**Contents**



**References.** E. Kamke. *Differentialgleichungen: Lösungsmethoden und Lösungen.*

## Preliminaries

The theory of ordinary differential equations (ODE) deals with the case when an unknown (differentiable) function depends on a unique variable and this function and its derivatives appear in a certain relation. For instance, the equation

$$y'' = xy' + (y')^2 \ , \tag{3.1}$$

where $y = y(x)$ is an unknown dependant variable and $x$ is an independent variable, is an ordinary differential equation (ODE) of order 2 (the maximal order of derivative is second : $y''$). To solve (3.1) means to find such functions $y = y(x)$ which turn (3.1) into identity (usually, on a certain interval $(a, b)$). Besides, the systems of ODE are also considered.

The theory of partial differential equations (PDE) deals the case when an unknown (differentiable) function depends on several variables and an equation involves the function and its partial derivatives (PDE are not subject of this topic).

Let us start with the following

**Example 3.1.** Find a plain curve $K$ with the following property:
*the segment of each tangent line to the curve $K$ contained between coordinate axes $Ox$ and $Oy$ is divided in two equal parts by the tangent point $T \in K$* (see Figure 3.1).

**Solution.** Suppose that such a curve $K$ is the plot of an unknown function $y = y(x)$. For simplicity's sake, consider the case $y > 0$, $x > 0$.

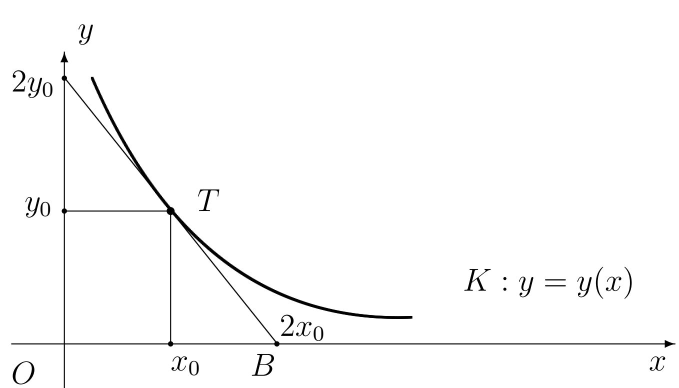


Fig. 3.1

Denote the coordinates of $T$ by $x_0$ and $y_0$. The equation of the tangent line is as follows:

$$y - y_0 = y'(x_0)(x - x_0)\ . \tag{3.2}$$

Since the segment of this tangent line concluded between coordinate axes $Ox$ and $Oy$ is divided in two equal parts by the tangent point $T = (x_0, y_0)$ then it is not hard to see that the point $B = (2x_0, 0)$ belongs to the tangent line.

It follows from (3.1) that the equality

$$0 - y_0 = y'(x_0)(2x_0 - x_0) \qquad \text{or}$$

$$y'(x_0) = -\frac{y_0}{x_0} \tag{3.3}$$

Since $x_0 > 0$ may be an arbitrary number then we obtain the differential equation

$$y' = -\frac{y}{x} \tag{3.4}$$

that describes such a curve $K$. Thus, we have made a first step, that is, we have composed a differential equation.

The second step is to solve the equation (3.4). For this purpose let us rewrite it in a slightly different form

$$\frac{dy}{dx} = -\frac{y}{x} \tag{3.5}$$

and note that the right-hand side splits: one factor depends only on $y$ and the other only on $x$.

Since we do not consider the simple case $y \equiv 0$ we may transform (3.5) into

$$\frac{dy}{y} = -\frac{dx}{x} \tag{3.6}$$

and then integrate:

$$\int \frac{dy}{y} = -\int \frac{dx}{x}\ . \tag{3.7}$$

It follows that

$$\ln|y| = -\ln|x| + \ln C\ ,$$

where $\ln C$ is an arbitrary constant. Hence

$$y = \pm\frac{C}{x} \quad \text{or}$$

$$y = \frac{C}{x}$$

if we do not consider $C$ as a *positive* constant. From the geometric point of view a curve $K$ is a hyperbola.

**Computations in MAPLE**

```
> ode := diff(y(x),x)+y(x)/x=0;
```

$$ode := \left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) + \frac{\mathrm{y}(x)}{x} = 0$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{\_C1}{x}$$

## 3.1. Ordinary differential equations of first order. Basic notions

**Definition 1.** *An equation*

$$F(x, y, y') = 0 \,, \tag{3.8}$$

*where $x$ is a variable, $y$, $y'$ are an unknown function (dependent variable) and its derivative, respectively, is called a differential equation of first order.*

Here are some examples of the ODE of first order:

$$(y')^3 y^2 + 2xy = 0 \,, \quad y = x^2 \cos y' \,, \quad (2xy + y^2)dy + 3x^2 dx = 0 \,.$$

The third equation can be transformed to the equation of first order as follows:

$$(2xy + y^2)dy + 3x^2 dx = 0$$

$$(2xy + y^2)dy = -3x^2 dx$$

$$\frac{dy}{dx} = -\frac{3x^2}{2xy + y^2} \quad \text{or} \quad y' = -\frac{3x^2}{2xy + y^2}$$

since

$$y' = \frac{dy}{dx} \,.$$

**Definition 2.** *An equation*

$$y' = f(x, y) \tag{3.9}$$

*is called a differential equation of first order resolved with respect to derivative.*

The equation (3.9) can be rewritten in the form

$$\frac{dy}{dx} = f(x, y)$$

or in the form

$$dy - f(x, y)dx = 0 \,.$$

Vice versa, the equation

$$M(x, y)dx + N(x, y)dy = 0$$

can be rewritten in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \quad \text{or} \quad y' = -\frac{M(x, y)}{N(x, y)} \,.$$

Thus, it can be regarded as an ODE of the type (3.9).

Here are some examples of ODE of the type (3.9):

$$y' = x^2 + y^2 \,, \quad y' = xy^3 + \operatorname{tg}(xy), \quad y' - y = 0 \,.$$

Consider two more examples:

$$\ln y' = y + x \Leftrightarrow y' = e^{x+y} \,,$$

$$(y')^2 = x^2 + y^2 \Leftrightarrow y' = \pm\sqrt{x^2 + y^2} \,.$$

In the last case we obtain two differential equations of first order.

**Definition 3.** *A function $y = y(x)$ is called a solution of the differential equation $F(x, y, y') = 0$ or $y' = f(x, y)$ if it satisfies the equation on some interval $(a, b)$ .*

The plot of the solution is called an *integral curve*. On Figure 3.1 you can see an integral curve $K$ of the equation (3.4). It is a branch of *hyperbola*.

**Example 3.2** The function $y = x^2$ makes trivial the left-hand side of the equation $xy' - 2x^2 = 0$ ($x \in \mathbb{R}$). Thus, $y = x^2$ is a solution of this ODE.

**The problem of existence and uniqueness** of the solution of an ODE is very important in the theory of ODE. A sufficient condition due to A.L. Cauchy provides an answer to this problem.

**Theorem 1.** *Given an equation $y' = f(x, y)$ let us suppose that $f(x, y)$ and $f'_y(x, y)$ are continuous functions in some domain $D$ of the plane $Oxy$. If $P_0 = (x_0, y_0)$ is an inner point of $D$ then locally there exists a unique solution $y = y(x)$ satisfying the condition $y_0 = y(x_0)$.* ("locally" means "in some neighborhood $U$ of $P_0$".)

**Initial conditions (IC).** The condition

$$y(x_0) = y_0 \quad \text{or} \quad y|_{x_0} = y_0$$

is called *initial* for a solution $y = y(x)$ for a differential equation of first order $y' = f(x, y)$. Due to Theorem 1 an initial condition provides existence and uniqueness of the corresponding integral curve. The problem

$$y' = f(x, y) \ , \quad y(x_0) = y_0 \ ,$$

that is, the problem to solve a given ODE of first order with a given initial condition, is called *Cauchy problem* or *initial condition (IC) problem*. Geometrically, to solve an IC problem means to find the integral curve passing through the prescribed point $(x_0, y_0)$.

**Definition 4.** *A function $y = y(x, C)$ is called the general solution of an ODE $y' = f(x, y)$ if it is a solution of this ODE for arbitrary (admissible) value of the constant $C$.*

For instance, $y = C/x$ is the general solution of the equation (3.4).

**Remark.** Sometimes it is more convenient to write the general solution either in a *parametric* form

$$\begin{cases} x = x(p, C) \\ y = y(p, C) \ , \end{cases}$$

where $p$ is a parameter, or in the *implicit* form

$$\Phi(x, y, C) = 0 \ .$$

For instance, the family $x^2 + y^2 = C$ is the general solution of the equation

$$y' = -\frac{x}{y} \ .$$

The general solution of this ODE can be written also as follows

$$\begin{cases} x = C \cos p \\ y = C \sin p \ . \end{cases}$$

Recall that for a parametric form

$$\frac{dy}{dx} = \frac{dy/dp}{dx/dp}$$

and for an implicit form

$$\frac{dy}{dx} = -\frac{\partial\Phi/\partial x}{\partial\Phi/\partial y} \ .$$

**The direction field for an ODE $y' = f(x,y)$ .** Let us suppose that we are given an ODE $y' = f(x,y)$ where $f(x,y)$ is a function defined for $(x,y) \in D$, $D \subseteq \mathbb{R}^2$. Let $y = y(x)$ be a solution of $y' = f(x,y)$ such that $y(x_0) = y_0$. It follows that the derivative

$$y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0) \ .$$

Consider the equation of the tangent line to the integral curve passing through the point $(x_0, y_0)$:

$$y - y_0 = y'(x_0)(x - x_0) \quad \text{or}$$

$$y - y_0 = f(x_0, y_0)(x - x_0) \ . \tag{3.10}$$

Thus, the line (3.10) is the tangent line to the integral curve (at the point $(x_0, y_0)$). This line can be drawn without solving the original differential equation. If we attach to each point $(x_0, y_0) \in D$ the line (3.10) (in fact, we need a sufficiently small segment of this line ) then we obtain the direction field for the equation $y' = f(x,y)$. The direction field approximates in some sense the integral curves in the domain $D$: at each point any integral curve is tangent to the line of the direction field.

**Example 3.2** Draw the direction field for the equation

$$y' = -\frac{x}{y} \ .$$

**Computations in MAPLE**

```
> with(DEtools):
> dfieldplot(diff(y(x),x)=-x/y(x),y(x),
> x=-2..2,
> y=-2..2, title=‘Restricted domain‘,color=1/4*(-(x^2+y^2)));
```

Note that the integral curves are circles $x^2 + y^2 = C$ (see Example 3.3 below) and compare with the obtained plot of the direction field.

## 3.2. Separable ODE of first order

**Definition 5.** *An equation*

$$y' = f_1(x) f_2(y) \tag{3.11}$$

*is called separable.*

The functions $f_1(x)$, $f_2(y)$ in (3.11) are supposed to be continuous.

The ODE of Example 3.1 is of the type (3.11). It is not hard to solve (3.11) using the algorithm of Example 3.1:

$$\frac{dy}{dx} = f_1(x) f_2(y) \ . \tag{3.12}$$

Further we assume that $f_2(y) \neq 0$. Then

$$\frac{dy}{f_2(y)} = f_1(x) dx \ . \tag{3.13}$$

and finally,

$$\int \frac{dy}{f_2(y)} = \int f_1(x) dx + C \ . \tag{3.14}$$

In (3.14) the variables are separated: $y$ is in the left-hand side of the equality and $x$ is in the right-hand side.

Thus, (3.14) represents the general solution of (3.11). Note that it is written in the implicit form.

**Example 3.3** Let us solve the initial condition (IC) problem

$$y' = -\frac{x}{y}, \quad y(1) = -2.$$

First of all, let us solve the ODE $y' = -\dfrac{x}{y}$.

$$\frac{dy}{dx} = -\frac{x}{y},$$

and, after the separation of variables,

$$ydy = -xdx,$$

$$\int ydy = -\int xdx.$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{C}{2}$$

and finally, $x^2 + y^2 = C$. Thus, all integral curves are circles centered in the origin $(0, 0)$.

The second step is to find the value of the constant $C$. Since $y_0 = -2$ if $x_0 = 1$ then $C = x_0^2 + y_0^2 = 1 + 4 = 5$. Hence, $x^2 + y^2 = 5$ is the desired solution. Also the function

$$y = -\sqrt{5 - x^2}$$

can be regarded as the solution of the IC problem.

**Computations in MAPLE**

```
> ode := diff(y(x),x)=-x/y(x);
```

$$ode := \frac{\partial}{\partial x}\,\mathrm{y}(x) = -\frac{x}{\mathrm{y}(x)}$$

```
> with(DEtools, odeadvisor);
> odeadvisor(ode);
```

$$[odeadvisor]$$

$$[\_separable]$$

```
> dsolve(ode,implicit);
```

$$\mathrm{y}(x)^2 + x^2 - \_C1 = 0$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \sqrt{-x^2 + \_C1},\ \mathrm{y}(x) = -\sqrt{-x^2 + \_C1}$$

```
> dsolve({ode,y(1)=-2});
```

$$\mathrm{y}(x) = -\sqrt{-x^2 + 5}$$

**Incomplete equations.** Equations

$$y' = f(x) \tag{3.15}$$

and

$$y' = g(y) \tag{3.16}$$

are called *incomplete*. The equation of the type (3.16) is also called *autonomous* because $g(y)$ does not involve the independent variable $x$ explicitly. The right-hand side of the equations depends on a unique variable, $x$ or $y$. These equations are of the type (3.11).

The general solution of (3.15) is

$$y = \int f(x)dx + C\,,$$

The general solution of (3.16) is

$$x + C = \int \frac{dy}{g(y)}$$

$(g(y) \neq 0)$.

**Example 3.4** Solve the ODE $y' = ky$ ($k$ is a constant).

Since $y \equiv 0$ is a solution we may consider the case when $y \neq 0$. Then

$$\frac{dy}{y} = kdx\,,$$

$$\int \frac{dy}{y} = k\int dx\,,$$

$$\ln|y| = kx + \ln C\,,$$

therefore,

$$y = Ce^{kx}\,. \tag{3.17}$$

Note that in general the constant $C$ in (3.17) may be trivial or negative.

**Computations in MAPLE**

```
>  ode := diff(y(x),x)=k*y(x);
>  dsolve(ode);
```

$$ode := \frac{\partial}{\partial x}\,\mathrm{y}(x) = k\,\mathrm{y}(x)$$

$$\mathrm{y}(x) = \_C1\, e^{(k\,x)}$$

**Example 3.5** Solve the ODE $y' = y^{2/3}$, $y(0) = 0$.

Since $y \equiv 0$ is a solution we may consider the case when $y \neq 0$. Then

$$\frac{dy}{y^{2/3}} = dx\,,$$

$$\int \frac{dy}{y^{2/3}} = \int dx\,,$$

$$3y^{1/3} = x + C\,,$$

$$y = \frac{(x+C)^3}{27}\,.$$

The initial condition provides $C = 0$ and

$$y = \frac{x^3}{27}$$

is another solution of the problem.

Why the solution of IC problem is not unique in this case ($y \equiv 0$ and $y = x^3/27$)? The derivative

$$(y^{2/3})'_y = \frac{2}{3y^{1/3}}$$

is not a continuous function at the point $(0, 0)$, thus, the conditions of Theorem 3.3 are not valid.

## 3.3. Homogeneous ODE of first order.

**Definition 6.** *An equation*

$$y' = f(y/x) \tag{3.18}$$

*is called homogeneous.*

For instance, the ODE

$$y' = y^2/x^2\ , \quad y' = \sin(x/y)\ , \quad y' = \frac{y^3 - 3xy^2}{x^2y + x^3}$$

are homogeneous ODE of first order. For the last one it follows from the presentation

$$y' = \frac{y^3 - 3xy^2}{x^2y + x^3} = \frac{(y^3 - 3xy^2)/x^3}{(x^2y + x^3)/x^3} = \frac{(y/x)^3 - 3(y/x)^2}{y/x + 1}\ .$$

In order to solve a homogeneous ODE $y' = f(y/x)$ one should make a substitution

$$p = \frac{y}{x}\ . \tag{3.19}$$

Here $p = p(x)$ is an unknown function as well as $y = y(x)$. Since $y = px$ then (3.18) turns into

$$y' = p'x + p = f(p)\ .$$

It follows that

$$p'x = f(p) - p\ , \quad \text{or}$$

$$p' = \frac{f(p) - p}{x}\ . \tag{3.20}$$

This is a separable ODE. The standard method of separation yields

$$\frac{dp}{dx} = \frac{f(p) - p}{x}\ ,$$

$$\int \frac{dx}{x} = \int \frac{dp}{f(p) - p}\ ,$$

$$\ln|x| = \int \frac{dp}{f(p) - p} + \ln C\ ,$$

$$x = C \exp \int \frac{dp}{f(p) - p}\ .$$

Since $y = px$ then

$$y = Cp\ \exp \int \frac{dp}{f(p) - p}$$

and we obtain the parametric solution

$$\begin{cases} x = C \cdot F(p) \\ y = Cp \cdot F(p)\ , \end{cases}$$

where $F(p)$ stands for $\exp \displaystyle\int \frac{dp}{f(p) - p}$ . Also note that the solution can be given in the implicit form

$$x = C \cdot F(y/x)\,.$$

**Example 3.6.** Solve the ODE

$$y' = \frac{y^2 + x^2}{xy}\,.$$

Substituting $y = px$ we obtain

$$p'x + p = p + \frac{1}{p} \quad \text{or}$$

$$\frac{dp}{dx} = \frac{1}{px}\,.$$

Therefore,

$$\int p\, dp = \int \frac{dx}{x}\,,$$

$$\frac{p^2}{2} = \ln|x| + C\,,$$

$$p = \pm\sqrt{2\ln|x| + 2C}\,,$$

and finally,

$$y = \pm\, x\,\sqrt{2\ln|x| + C_1}\,,$$

where $C_1 = 2C$.

**Computations in MAPLE**

```
> ode := diff(y(x),x)=((y(x))^2+x^2)/(y(x)*x);
> dsolve(ode);
```

$$ode := \frac{\partial}{\partial x}\,\mathrm{y}(x) = \frac{\mathrm{y}(x)^2 + x^2}{\mathrm{y}(x)\,x}$$

$$\mathrm{y}(x) = \sqrt{2\ln(x) + \_C1}\; x,\ \mathrm{y}(x) = -\sqrt{2\ln(x) + \_C1}\; x$$

```
> with(DEtools, odeadvisor);
> odeadvisor(ode);
```

$$[odeadvisor]$$

$$[[\_homogeneous,\ class\ A],\ \_rational,\ \_Bernoulli]$$

## 3.4. Linear ODE of first order

**Definition 7.** *An equation*

$$y' + p(x)y = q(x) \tag{3.21}$$

*is called linear first-order ODE.*

If $q(x) \equiv 0$ then the equation (3.21) is called *linear homogeneous*, otherwise it is called *linear non-homogeneous.*

To solve (3.21) one can either make a Bernoulli's substitution or use the variation of the constant method (these two ways do not differ too much from the mathematical point of view).

**Bernoulli's substitution.** Let us look for a solution of the linear ODE (3.21) as a product

$$y = uv \tag{3.22}$$

where $u = u(x)$ and $v = v(x)$ are two unknown functions. If we make a substitution (3.22) following Bernoulli then we obtain

$$u'v + uv' + p(x)uv = q(x) \quad \text{or}$$

$$u'v + u \cdot (v' + p(x)v) = q(x) \tag{3.23}$$

**Step 1.** Now suppose that $V = V(x)$ is a non-trivial particular solution of the equation

$$v' + p(x)v = 0\ . \tag{3.24}$$

Note that the equation (3.24) can be regarded either as the linear homogeneous ODE associated to the equation (3.21) or as a separable equation. Thus, it is not hard to write down such a solution, for instance,

$$V(x) = \exp\left(-\int e^{p(x)}dx\right)\ . \tag{3.25}$$

**Step 2.** In view of (3.23) and (3.24) we reduce (3.23) to the form

$$u' \cdot V(x) = q(x)\ .$$

Since $V(x) \neq 0$ we obtain

$$u' = \frac{q(x)}{V(x)}$$

and finally

$$u = \int \frac{q(x)}{V(x)}\, dx + C = \int q(x) \cdot \exp\left(\int e^{p(x)}dx\right)\, dx + C\ .$$

It follows that

$$y = uv = C\exp\left(-\int e^{p(x)}dx\right) + \exp\left(-\int e^{p(x)}dx\right) \cdot \int q(x) \cdot \exp\left(\int e^{p(x)}dx\right)\, dx$$

or, in view of (3.25),

$$y = C \cdot V(x) + V(x) \cdot \int \frac{q(x)}{V(x)}\, dx\ . \tag{3.26}$$

**Remark.** Note that we have obtained the general solution in the form

$$y = y_{gh} + y_{pn} \tag{3.27}$$

where

$$y_{gh} = C \cdot V(x)$$

is the general solution of the homogeneous equation (3.24) ('$g$' stands for "general" and '$h$' for "homogeneous" ) and

$$y_{pn} = V(x) \cdot \int \frac{q(x)}{V(x)}\, dx$$

is a particular solution of the non-homogeneous equation (3.21).

**Method of variation of constant.** This method for solving a linear ODE (3.21) also consists of two steps.

**Step 1.** We solve the auxiliary homogeneous equation (3.24) and obtain

$$v(x) = C \cdot V(x)\ , \tag{3.28}$$

$V(x)$ is given by the formula (3.25).

**Step 2.** Variation of the constant $C$.

In view of (3.28) we search the unknown function $y$ in the form

$$y = y(x) = C(x) \cdot V(x)\ , \tag{3.29}$$

where $C(x)$ is a certain function depending on $x$. Since $y$ must be a solution of (3.21) we get substituting $y$ from (3.29) into the ODE (3.21)

$$C'(x) \cdot V(x) + C(x) \cdot V'(x) + p(x)\, C(x) V(x) = q(x)\,,$$

or

$$C'(x) \cdot V(x) + C(x)\, (V'(x) + p(x)\, V(x)) = q(x)\,.$$

The function $V(x)$ is a particular solution of the homogeneous ODE (3.24), hence $V'(x) + p(x)\, V(x) = 0$. Now we obtain the equation

$$C'(x) \cdot V(x) = q(x)$$

with the known function $V(x)$. Then

$$C(x) = \int \frac{q(x)}{V(x)}\, dx + C\ .$$

In view of (3.29) we obtain the general solution given by the formula (3.26).

**Example 3.7.** Solve the equation

$$y' + \frac{2y}{x} = x^2 + 1\ .$$

First of all, we substitute $y$ by the product $uv$ and get

$$u'v + uv' + \frac{2uv}{x} = x^2 + 1\ , \quad \text{or}$$

$$u'v + u\left(v' + \frac{2v}{x}\right) = x^2 + 1\ . \tag{3.30}$$

**Step 1.** The equation

$$v' + \frac{2v}{x} = 0$$

is separable, then

$$\frac{dv}{dx} = -\frac{2v}{x} \,,$$

$$\int \frac{dv}{v} = -\int \frac{2dx}{x} \,,$$

$$\ln|v| = -2\ln|x| = \ln\frac{1}{x^2} \,.$$

We omit a constant of integration since we need only a non-trivial *particular* solution of the equation. Hence, $V(x) = \dfrac{1}{x^2}$ is such a solution.

**Step 2.** Coming back to (3.30) we get

$$u' \cdot \frac{1}{x^2} = x^2 + 1 \,, \quad \text{or}$$

$$u' = x^4 + x^2 \,,$$

whence

$$u = \int (x^4 + x^2)\, dx = \frac{x^5}{5} + \frac{x^3}{3} + C \,.$$

Thus,

$$y = uv = \left(\frac{x^5}{5} + \frac{x^3}{3} + C\right) \cdot \frac{1}{x^2} = \frac{x^3}{5} + \frac{x}{3} + \frac{C}{x^2}$$

is the general solution. Note that

$$y_{gh} = \frac{C}{x^2}$$

is the general solution of the linear homogeneous equation and

$$y_{pn} = \frac{x^3}{5} + \frac{x}{3}$$

is a particular solution of the non-homogeneous equation.

**Computations in MAPLE**

```
> ode := diff(y(x),x)+2*y(x)/x=x^2+1;
```

$$ode := \left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) + \frac{2\,\mathrm{y}(x)}{x} = x^2 + 1$$

```
> with(DEtools, odeadvisor):
> odeadvisor(ode);
```

$$[\_linear]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{1}{5}\,x^3 + \frac{1}{3}\,x + \frac{\_C1}{x^2}$$

## 3.5. ODE of Bernoulli type

**Definition 8.** *An equation*

$$y' + p(x)y = q(x)y^n \tag{3.31}$$

*is called Bernoulli's ODE.*

To solve (3.31) one can either make a Bernoulli substitution as earlier or use the power substitution $z = y^{1-n}$ in order to reduce the equation (3.31) to a linear ODE.

**Bernoulli's substitution.** We search a solution of the ODE (3.31) again as a product

$$y = uv\,.$$

In view of (3.31) we obtain

$$u'v + uv' + p(x)uv = q(x)u^n v^n \quad \text{or}$$

$$u'v + u\cdot(v' + p(x)v) = q(x)u^n v^n\ . \tag{3.32}$$

**Step 1.** It is the same as in solving linear ODE. We find $V = V(x)$ which is a non-trivial particular solution of the separable equation $v' + p(x)v = 0$ , for instance,

$$V(x) = \exp\left(-\int e^{p(x)}dx\right)\ .$$

**Step 2.** In view of (3.32) and (3.24) we reduce (3.32) to the form

$$u'\cdot V(x) = q(x)V(x)^n \cdot u^n$$

with known $V(x)$. Since $V(x) \neq 0$ then we get

$$u' = q(x)V^{n-1}(x)\cdot u^n$$

which is a separable ODE needed to be solved. Hence,

$$\frac{du}{u^n} = q(x)V^{n-1}(x)\,dx\ ,$$

and after the integration of both parts the function $u$ can be found since

$$\int\frac{du}{u^n} = -\frac{1}{(n-1)u^{n-1}} \qquad (n\neq 1)\,.$$

If $n = 1$ then (3.31) is a linear homogeneous ODE.

Now it remains to present the general solution in the form $y = uV(x)$.

**Example 3.8.** Solve the equation

$$y' + xy = x^3y^3\ .$$

Making the Bernoulli's substitution (3.22) we obtain

$$u'v + u(v' + vx) = x^3u^3v^3\ . \tag{3.33}$$

**Step 1.** The equation $v' + vx = 0$ is separable. Then

$$\frac{dv}{dx} = -vx\ , \qquad \int\frac{dv}{v} = -\int x\,dx\ , \qquad \ln|v| = -\frac{x^2}{2}\ .$$

Hence, $V(x) = e^{-x^2/2}$.

**Step 2.** Coming back to (3.33) we obtain

$$u' \, e^{-x^2/2} = x^3 \, e^{-3x^2/2} \cdot u^3 \,,$$

or

$$\frac{du}{dx} = x^3 \, e^{-x^2} \cdot u^3 \,.$$

Then

$$\int \frac{du}{u^3} = \int x^3 \, e^{-x^2} \, dx \,.$$

Hence,

$$-\frac{1}{2u^2} = -\frac{1}{2}(x^2+1)\, e^{-x^2} - \frac{C}{2} \quad \text{or}$$

$$u = \pm \frac{1}{\sqrt{C + (x^2+1)e^{-x^2}}} \,.$$

It follows that

$$y = uV(x) = \pm \frac{e^{-x^2/2}}{\sqrt{C + (x^2+1)e^{-x^2}}} = \pm \frac{1}{\sqrt{Ce^{x^2} + x^2 + 1}} \,.$$

**Computations in MAPLE**

```
> ode := diff(y(x),x)+x*y(x)=x^3*y(x)^3;
```

$$ode := (\tfrac{\partial}{\partial x}\,\mathrm{y}(x)) + x\,\mathrm{y}(x) = x^3\,\mathrm{y}(x)^3$$

```
> with(DEtools, odeadvisor):
> odeadvisor(ode);
```

$$[\_Bernoulli]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{1}{\sqrt{x^2 + 1 + e^{(x^2)}\,\_C1}},\ \mathrm{y}(x) = -\frac{1}{\sqrt{x^2 + 1 + e^{(x^2)}\,\_C1}}$$

## 3.6. Exact ODE. The integrating factor

**Definition 9.** *An equation of first order in the form*

$$P(x,y)dx + Q(x,y)dy = 0 \tag{3.34}$$

*is called exact ODE ($P = P(x,y)$ and $Q = Q(x,y)$ are supposed to be differentiable functions) if the condition*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{3.35}$$

*is satisfied.*

To solve an exact ODE (3.34) one should find such a function $u = u(x,y)$ such that

$$\frac{\partial u}{\partial x} = P(x,y)\,, \qquad \frac{\partial u}{\partial y} = Q(x,y)\,.$$

A function $u$ exists if $(x, y)$ runs over a $1-$*connected* plain domain $D \subseteq \mathbb{R}^2$ (for instance, $D$ is a plane, half-plane, circle etc. $D$ must have "no hole" inside).

Then the implicit form of the general solution is

$$u(x, y) = C \ .$$

**Example 3.9.** Solve the equation

$$\left(2x + y^3 e^{xy^3}\right) dx + 3xy^2 e^{xy^3} dy = 0 \ .$$

Here

$$P = 2x + y^3 e^{xy^3} \ , \quad Q = 3xy^2 e^{xy^3} \ ; \qquad (x, y) \in \mathbb{R}^2.$$

The condition (3.35) is satisfied since

$$\frac{\partial P}{\partial y} = 3y^2 e^{xy^3} + 3y^5 e^{xy^3} = \frac{\partial Q}{\partial x} \ .$$

Let us find $u = u(x, y)$. For this purpose solve the system

$$\frac{\partial u}{\partial x} = P = 2x + y^3 e^{xy^3} \ , \qquad \frac{\partial u}{\partial y} = Q = 3xy^2 e^{xy^3} \ .$$

The integration of the second equation yields the preliminary form

$$u = \int 3xy^2 e^{xy^3} dy = e^{xy^3} + C(x) \qquad (3.36)$$

where $C(x)$ is a differentiable function depending only on $x$.

From the one hand (see the first equation of the system)

$$P = \frac{\partial u}{\partial x} = 2x + y^3 e^{xy^3} \ .$$

From the other hand (it follows from (3.36))

$$P = \frac{\partial u}{\partial x} = y^3 e^{xy^3} + C'(x) \ .$$

By comparison $C'(x) = 2x$ and

$$C(x) = x^2$$

is a particular function $C(x)$.

We get from (3.36) that

$$u = e^{xy^3} + x^2$$

is a desired function in variables $x$, $y$. The general solution can be written in the implicit form

$$e^{xy^3} + x^2 = C \ .$$

**Computations in MAPLE**

First, we rewrite the equation as follows

$$\left(2x + y^3 e^{xy^3}\right) + 3xy^2 e^{xy^3} \frac{dy}{dx} = 0 \ .$$

```
> ode:=diff(y(x),x)*(3*x*y(x)^2*exp(x*y(x)^3))+(2*x+y(x)^3*exp(x*y(x)^3
> ))=0;
```

$$ode := 3\,(\frac{\partial}{\partial x}\,\mathrm{y}(x))\,x\,\mathrm{y}(x)^2\,e^{(x\,\mathrm{y}(x)^3)} + 2\,x + \mathrm{y}(x)^3\,e^{(x\,\mathrm{y}(x)^3)} = 0$$

```
> with(DEtools, odeadvisor):
> odeadvisor(ode);
```

$$[\_exact, [\_1st\_order, \_with\_symmetry\_[F(x), G(x)*y+H(x)]]]$$

```
> dsolve(ode,implicit);
```

$$x^2 + e^{(x\,\mathrm{y}(x)^3)} + \_C1 = 0$$

**The integrating factor**

Suppose that we are given an ODE

$$L(x,y)dx + M(x,y)dy = 0$$

which is not exact but the equation

$$\mu(x,y)\cdot L(x,y)dx + \mu(x,y)\cdot M(x,y)dy = 0$$

is exact. Such a factor $\mu = \mu(x,y)$ is called an *integrating* factor.

Consider the following example.

**Example 3.10.** Find the integrating factor for the equation

$$\left(3x^2y + y^2\right)dx + (2x^3 + 3xy)dy = 0\ .$$

Here

$$L = 3x^2y + y^2\ ,\quad M = 2x^3 + 3xy$$

and

$$\frac{\partial L}{\partial y} = 3x^2 + 2y \neq 6x^2 + 3y = \frac{\partial M}{\partial x}\ ,$$

that is, the equation is not exact.

In general, it is not easy to find the integrating factor if it depends on $x$ and $y$ (it means just to solve a given equation). However, if $\mu = \mu(x)$ or $\mu = \mu(y)$ then $\mu$ can be found from an appropriate differential equation.

We continue the investigation of Example 3.10. Let us suppose that $\mu = \mu(y)$ does not depend on $x$. Then the condition of exactness (3.35) implies

$$(\mu L)'_y = (\mu M)'_x$$

or

$$\mu'_y L + \mu L'_y = \mu M'_x\ .$$

It follows that

$$\mu'_y = \frac{\mu(M'_x - L'_y)}{L} = \frac{\mu(6x^2 + 3y - 3x^2 - 2y)}{3x^2y + y^2} = \frac{\mu}{y}\ .$$

This separable ODE has a particular solution $\mu = y$. Thus, the integrating factor is equal to $y$.

**Computations in MAPLE**

We rewrite the equation as follows

$$3x^2y + y^2 + (2x^3 + 3xy)\frac{dy}{dx} = 0 \ .$$

```
> ode := diff(y(x),x)*(2*x^3+3*x*y(x))+(3*x^2*y(x)+y(x)^2)=0;
```

$$ode := (\frac{\partial}{\partial x}\,\mathrm{y}(x))\,(2\,x^3 + 3\,x\,\mathrm{y}(x)) + 3\,x^2\,\mathrm{y}(x) + \mathrm{y}(x)^2 = 0$$

```
> with(DEtools):
> intfactor(ode);
```

$$\mathrm{y}(x)$$

```
> dsolve(ode,implicit);
```

$$\ln(x) - \_C1 + \frac{2}{7}\ln(\frac{\mathrm{y}(x)}{x^2}) + \frac{1}{7}\ln(\frac{x^2 + \mathrm{y}(x)}{x^2}) = 0$$

## 3.7. First order equations unresolved with respect to derivative

### 3.7.1. Equation $y = f(x, y')$.

The equation

$$y = f(x, y') \tag{3.37}$$

is resolved with respect to the unknown function $y = y(x)$. In order to solve this equation let us introduce the notation $p = y'$. Then we get the system

$$\begin{cases} y = f(x, p) \\ p = y' \end{cases}$$

Differentiating the first equation by $x$ we get

$$y' = f'_x(x, p) + f'_y(x, p)p' \ ,$$

or, since $y' = p$

$$p = f'_x(x, p) + f'_y(x, p)p' \ .$$

Hence,

$$p' = \frac{p - f'_x(x, p)}{f'_y(x, p)} \tag{3.38}$$

is a first-order ODE resolved with respect to the derivative $p'$.

Let us suppose that $p = p(x, C)$ is the general solution of (3.38). Then in view of the equation $y = f(x, p)$ we can obtain the general solution of the ODE (3.37):

$$y = f(x, p(x, C)) \ .$$

**Clairaut equation.** Solve the equation

$$y = xy' - f(y') \tag{3.39}$$

where $f$ is a given differentiable function.

The equation (3.39) is known as *Clairaut equation*. Let $p = y'$ as above. Then

$$y = xp - f(p)$$

and

$$y' = p + xp' - f'(p)p' \quad \text{or} \quad p = p + xp' - f'(p)p' .$$

It follows that

$$p'(x - f'(p)) = 0 .$$

Now two cases occur.

**Case 1.** Let $p' = 0$. It means that $p = C$ where $p = y'$. In view of (3.39)

$$y = xC - f(C) . \tag{3.40}$$

This is the general solution of Clairoaut equation.

**Case 2.** Let $x = f'(p)$. In view of (3.39) $y = pf'(p) - f(p)$. Thus we obtain the parametric solution

$$\begin{cases} x = f'(p) \\ y = pf'(p) - f(p) . \end{cases} \tag{3.41}$$

The last solution is called the *special* solution of the Clairaut equation.

**Remark.** It is not hard to see that the tangent line to the integral curve corresponding to the special solution coincides with a line given by (3.40) for an appropriate constant $C$. It follows that each point of special integral curve belongs to another integral curve (line). Thus, the property of uniqueness is not true.

**Example 3.11.** Solve the equation

$$y = xy' - (y')^2 .$$

Let $p = y'$. Then $y = xp - p^2$ and

$$p = y' = (xp - p^2)' = p + xp' - 2pp' .$$

Hence,

$$(x - 2p)p' = 0 .$$

**Case 1.** Let $p' = 0$. Then $p = C$ and

$$y = xC - C^2$$

is the general solution.

**Case 2.** Let $x = 2p$. Then $y = 2p \cdot p - p^2 = p^2$,

$$\begin{cases} x = 2p \\ y = p^2 , \end{cases}$$

is the special solution.

Note the the special integral curve is the parabola $\mathcal{P} : y = x^2/4$ and the family $y = xC - C^2$ consists of tangent lines to $\mathcal{P}$.

**Computations in MAPLE**

```
> ode := y(x)=x*diff(y(x),x)-diff(y(x),x)^2;
```

$$ode := \mathrm{y}(x) = x\,(\tfrac{\partial}{\partial x}\,\mathrm{y}(x)) - (\tfrac{\partial}{\partial x}\,\mathrm{y}(x))^2$$

```
> with(DEtools, odeadvisor):
> odeadvisor(ode);
```

$$[[\_1st\_order,\ \_with\_linear\_symmetries],\ \_Clairaut]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{1}{4}\,x^2,\ \mathrm{y}(x) = x\,\_C1 - \_C1^2$$

```
> plot([x^2/4,x-1,2*x-4], x=-1..6,color=[blue,red,black]);
```

**3.7.2. Equation $x = f(y, y')$.**

The equation

$$x = f(y, y') \tag{3.42}$$

is resolved with respect to the independent variable $x$. Let us again introduce the notation $p = y'$. Then we get the system

$$\begin{cases} x = f(y, p) \\ p = y' \end{cases}$$

Differentiating the first equation by $y$ we get

$$\frac{dx}{dy} = f'_y(y, p) + f'_p(y, p) \cdot \frac{dp}{dy}\ ,$$

or, since $\dfrac{dx}{dy} = \dfrac{1}{p}$,

$$\frac{dp}{dy} = \frac{1/p - f'_y(y, p)}{f'_p(y, p)}\ . \tag{3.43}$$

It is a first-order ODE resolved with respect to the derivative $dp/dy$.

Suppose that $p = p(y, C)$ is the general solution of (3.43). Then in view of the equation $x = f(y, p)$ we can obtain the general solution of the ODE (3.42):

$$x = f(y, p(y, C))\,.$$

## 3.8. ODE of second order. Examples

**Definition 10.** An equation

$$F(x, y, y', y'') = 0 \tag{3.44}$$

is called *ODE of second order.*

**Examples.** The following ODE

$$y'' = -\omega^2 y$$

where $\omega$ is some (positive) constant provides an example of such an ODE. This ODE is called *the equation of mathematical pendulum (oscillator)*. It describes the small oscillations of a pendulum.

The ODE

$$y'' = -A \sin y\ , \qquad A > 0,$$

is called *the equation of physical pendulum.*

The two equations are resolved with respect to the derivative of second order. In general, an equation

$$y'' = f(x, y, y') \tag{3.45}$$

is called a second order ODE resolved with respect to the second derivative of unknown function.

**Definition 11.** *A function $y = y(x)$ is called a solution of the differential equation $F(x, y, y', y'') = 0$ or $y'' = f(x, y, y')$ if it satisfies the equation on some interval $(a, b)$ .*

The plot of the solution is called an *integral curve.*

**Example 3.12.** The function $y = C_1 \cos \omega x + C_2 \sin \omega x$ is a solution of the equation $y'' = -\omega^2 y$.

**The problem of existence and uniqueness** can be resolved positively under certain conditions given by the following theorem.

**Theorem 2.** *Given an equation $y'' = f(x, y, y'')$ suppose that $f(x, y, y')$ , $f'_y(x, y, y')$ and $f'_{y'}(x, y, y')$ are continuous functions in some domain $D$ of the space of variables. If $P_0 = (x_0, y_0, y_1)$ is an inner point of $D$ then locally there exists a unique solution $y = y(x)$ satisfying the conditions $y(x_0) = y_0$, $y'(x_0) = y_1$ .*

**Initial conditions (IC).** The conditions

$$y(x_0) = y_0 \, , \; y'(x_0) = y_1$$

are called *initial* for a solution $y = y(x)$ of a second order differential equation $y'' = f(x, y, y')$. The problem

$$y'' = f(x, y, y') \, , \quad y(x_0) = y_0 \, , \; y'(x_0) = y_1 \, ,$$

that is, the problem to solve a given ODE of second order with a given two initial conditions, is called *Cauchy problem* or *initial condition problem.*

**Definition 12.** *A function $y = y(x, C_1, C_2)$ is called general solution of an ODE $y'' = f(x, y, y')$ if it is a solution of this ODE for arbitrary value of constants $C_1$, $C_2$.*

Note that usually two constants $C_1$, $C_2$ can be found from two initial conditions.

**Example 3.13.** Solve the initial condition problem

$$y'' = -\omega^2 y \, , \quad y(0) = y_0 \, , \; y'(0) = v_0$$

for the pendulum equation.

In the pendulum equation the variable $y$ means the distance from the equilibrium point ($y = 0$), $x$ means the time and $y'$ means the velocity (see Figure 3.2). Hence, $y(0) = y_0$ means the initial distance from the equilibrium point and $y'(0) = v_0$ means the initial velocity.

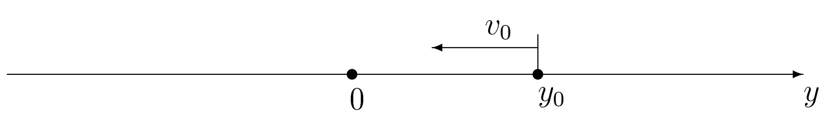


Figure 3.2

It is not hard to check that the general solution of the equation $y'' = -\omega^2 y$ is

$$y = C_1 \cos \omega x + C_2 \sin \omega x \, .$$

Hence, $\omega$ is the frequency of oscillations.

By the first initial condition,

$$y_0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 \; .$$

After differentiation we have $y' = -C_1\omega \sin\omega x + C_2\omega\cos\omega x$ whence

$$v_0 = y'(0) = -C_1\omega\sin 0 + C_2\omega\cos 0 = C_2\omega\,.$$

It follows that

$$y = y_0\,\cos\omega x + \frac{v_0}{\omega}\,\sin\omega x$$

is the solution of the initial condition problem. From the physical point of view this function represents the periodic dependance of the coordinate $y$ on the time $x$ (in other words, an oscillation).

**Computations in MAPLE**

```
> with(DEtools):
```

```
> ode := diff(y(x),x$2)=-omega^2*y(x);
```

$$ode := \frac{\partial^2}{\partial x^2}\,\mathrm{y}(x) = -\omega^2\,\mathrm{y}(x)$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \_C1\,\sin(\omega\,x) + \_C2\,\cos(\omega\,x)$$

```
> dsolve({ode, y(0)=y0, D(y)(0)=v0}, y(x));
```

$$\mathrm{y}(x) = \frac{v0\,\sin(\omega\,x)}{\omega} + y0\,\cos(\omega\,x)$$

**Boundary conditions (BC).** The conditions

$$y(x_0) = y_0\,,\ y(x_1) = y_1$$

are called *boundary* for a solution $y = y(x)$ of the second order differential equation $y'' = f(x, y, y')$. The problem

$$y'' = f(x, y, y')\,,\quad y(x_0) = y_0\,,\ y(x_1) = y_1\,,$$

that is, the problem to solve a given ODE of second order with prescribed two values of $y$ at points $x_0$, $x_1$, is called *boundary condition problem.*

## 3.9. Linear ODE of second order with constant coefficients

**Definition 11.** The following equation

$$y'' + py' + qy = f(x) \tag{3.46}$$

is called *linear ODE of second order with constant coefficients* provided $p, q$ are two given constants.

An equation

$$y'' + py' + qy = 0 \tag{3.47}$$

is called *linear homogeneous ODE with constant coefficients*. For instance, the equation $y'' = -\omega^2 y$ is homogeneous.

### 3.9.1. Linear homogeneous equations

Let us look for a solution of a given ODE (3.47) in the form $y = e^{kx}$ ($k$ is some constant). Substituting the derivatives of $y$ into (3.47) we obtain the following equality

$$k^2 \cdot e^{kx} + pk \cdot e^{kx} + q \cdot e^{kx} = 0 \, .$$

Reducing by $e^{kx}$ we get a quadratic equation

$$k^2 + pk + q = 0 \, . \tag{3.48}$$

This equation is called *characteristic* and the quadratic polynomial $P(k) = k^2 + pk + q$ is called *characteristic polynomial* for the equation (3.47). Then the form of a solution of (3.47) depends on the roots of characteristic polynomial (and the multiplicities of roots).

One can prove the following theorem.

**Theorem 3.**
**1.** If the roots $k_1$, $k_2$ of the characteristic polynomial are real ($k_1 \in \mathbb{R}$, $k_2 \in \mathbb{R}$) and distinct ($k_1 \neq k_2$) then the general solution of the ODE (3.47) is

$$y = C_1 \, e^{k_1 x} + C_2 \, e^{k_2 x} \, .$$

In this case $P(k) = k^2 + pk + q = (k - k_1)(k - k_2)$.
**2.** If the roots $k_1$, $k_2$ of the characteristic polynomial are real and coincide ($k_1 = k_2 \in \mathbb{R}$) then the general solution of the ODE (3.47) is

$$y = (C_1 + C_2 x) \, e^{k_1 x} \, .$$

In this case $P(k) = k^2 + pk + q = (k - k_1)^2$.
**3.** If the roots $k_1$, $k_2$ of the characteristic polynomial are complex ($k_1 = a + bi$, $k_2 = a - bi$ where $i = \sqrt{-1}$) then the general solution of the ODE (3.47) is

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx \, .$$

In this case $P(k) = k^2 + pk + q = (k - k_1)(k - k_2)$.
**Example 3.14.** Solve the equation $y'' + 5y' + 4y = 0$ .

The characteristic equation is $k^2 + 5k + 4 = 0$ or $(k + 1)(k + 4) = 0$. Hence, $k_1 = -1$, $k_2 = -4$. It follows that

$$y = C_1 \, e^{-x} + C_2 \, e^{-4x}$$

is the general solution of the given equation.
**Example 3.15.** Solve the equation $y'' - 6y' + 9y = 0$ .

The characteristic equation is $k^2 - 6k + 9 = 0$ or $(k - 3)^2 = 0$. Hence, $k_1 = k_2 = 3$ is the unique root of multiplicity 2. It follows that

$$y = (C_1 + C_2 x) \, e^{3x}$$

is the general solution of the given equation.
**Example 3.16.** Solve the equation $y'' - 6y' + 13y = 0$ .

The characteristic equation is $k^2 - 6k + 13 = 0$. We have

$$k_{1,2} = 3 \pm \sqrt{3^2 - 13} = 3 \pm 2i \, .$$

Thus, the general solution is

$$y = C_1 e^{3x} \cos 2x + C_2 e^{3x} \sin 2x \, .$$

**Example 3.17.** Solve the equation $y'' = -\omega^2 y$ .

The characteristic equation is $k^2 + \omega^2 = 0$. The roots are

$$k_{1,2} = \pm\,\omega\, i\ .$$

The general solution is

$$y = C_1 \cos \omega x + C_2 \sin \omega x\ .$$

**Computations in MAPLE**

```
> with(DEtools):
> ode1 := diff(y(x),x$2)+5*diff(y(x),x)+4*y(x)=0;
```

$$ode1 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + 5\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 4\,\mathrm{y}(x) = 0$$

```
> dsolve(ode1);
```

$$\mathrm{y}(x) = \_C1\, e^{(-4\,x)} + \_C2\, e^{(-x)}$$

```
> ode2 := diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=0;
```

$$ode2 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) - 6\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 9\,\mathrm{y}(x) = 0$$

```
> dsolve(ode2);
```

$$\mathrm{y}(x) = \_C1\, e^{(3\,x)} + \_C2\, e^{(3\,x)}\, x$$

```
> ode3 := diff(y(x),x$2)-6*diff(y(x),x)+13*y(x)=0;
```

$$ode3 := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) - 6\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 13\,\mathrm{y}(x) = 0$$

```
> dsolve(ode3);
```

$$\mathrm{y}(x) = \_C1\, e^{(3\,x)} \sin(2\,x) + \_C2\, e^{(3\,x)} \cos(2\,x)$$

```
> P1:=k^2+5*k+4;
```

$$P1 := k^2 + 5\,k + 4$$

```
> factor(P1);
```

$$(k+4)\,(k+1)$$

```
> P2:=k^2-6*k+9;
```

$$P2 := k^2 - 6\,k + 9$$

```
> factor(P2);
```

$$(k-3)^2$$

```
> P3:=k^2-6*k+13;
```

$$P3 := k^2 - 6\,k + 13$$

```
> factor(P3,complex);
```

$$(k - 3.000000000 + 2.000000000\,I)\,(k - 3. - 2.000000000\,I)$$

### 3.9.2. Linear non-homogeneous equations

Consider a linear non-homogeneous equation of second order

$$y'' + py' + qy = f(x)\ .$$

It is not hard to see that the general solution of this equation can be represented in the form

$$y = y_{gh} + y_{pn}$$

where $y_{gh}$ is the general solution of the linear homogeneous equation $y'' + py' + qy = 0$ with the same coefficients $p$, $q$ as above and $y_{pn}$ is a particular solution of the given non-homogeneous equation.

In the previous subsection we explained how to find the general solution of the linear homogeneous equation $y_{gh}$. Thus it remains to find the second term $y_{pn}$. Further we will be concerned with the case when the right-hand term of the equation $f(x)$ is a so called *quasi-polynomial.*

By definition, $f(x)$ is a quasi-polynomial if it a function either of the form

$$L(x)\, e^{ax} \qquad \text{(type I)}$$

or of the form

$$L(x)\, e^{ax} \cos bx + M(x)\, e^{ax} \sin bx \qquad \text{(type II)}$$

where $L(x)$, $M(x)$ are some fixed polynomials (for type II it is possible that $L(x)$ or $M(x)$ is trivial) and $a$, $b \neq 0$ are real constants.

**I**. If $f(x) = L(x)\, e^{ax}$ then, by definition, the degree $d$ of $f(x)$ is the degree of polynomial $L(x)$ and the exponent $a$ is called the exponent of $f(x)$.

**II.** If $PL(x)\, e^{ax} \cos bx + M(x)\, e^{ax} \sin bx$ then the degree $d$ of $f(x)$ is the maximal degree of polynomials $L(x)$, $M(x)$ and the pair of complex numbers $\{a \pm bi\}$ is called the pair of exponents for $f(x)$.

The following theorem can be proved.

**Theorem 4.** In previous notation
**1.** If the quasi-polynomial $f(x)$ has the exponent $a$ (or the pair of exponents $\{a \pm bi\}$ for $f(x)$ of the type II) which is not a root of the characteristic polynomial $P(k) = k^2 + pk + q$ then there exists a particular solution $y_{pn}$ which is a quasi-polynomial of the same degree and the same exponent as $f(x)$ (no resonance).
**2.** If the quasi-polynomial $f(x)$ has the exponent $a$ (or the pair of exponents $\{a \pm bi\}$ for $f(x)$ of the type II) which is a root of multiplicity $m$ of the characteristic polynomial $P(k) = k^2 + pk + q$ then there exists a particular solution $y_{pn}$ which is a quasi-polynomial of the same degree and the same exponent as $f(x)$ multiplied by $x^m$ (resonance of order $m$).

**Example 3.18.** Consider the equation

$$y'' - 4y' + 3y = (2x + 3)e^{5x}\,.$$

Here $f(x) = (2x + 3)e^{5x}$ is a quasi-polynomial of degree $d = 1$ and of the exponent $a = 5$. Since $a = 5$ is not a root of $P(k) = k^2 - 4k + 3$ then we have no resonance and there is a particular solution of the form

$$y_{pn} = (Ax + B)e^{5x}\,.$$

Let us find the coefficients $A$, $B$. Since

$$y'_{pn} = Ae^{5x} + 5(Ax + B)e^{5x} = (5Ax + 5B + A)e^{5x}\,,$$
$$y''_{pn} = 5Ae^{5x} + 5(5Ax + 5B + A)e^{5x} = (25Ax + 25B + 10A)e^{5x}$$

then

$$y''_{pn} - 4y'_{pn} + 3y_{pn} = (8Ax + 6A + 8B)e^{5x}\,.$$

**Computations in MAPLE**

```
> z:=x->(A*x+B)*exp(5*x);
```

$$z := x \to (A\,x + B)\,e^{(5\,x)}$$

```
> diff(z(x),x$2)-4*diff(z(x),x)+3*z(x);
```

$$6\,A\,e^{(5\,x)} + 8\,(A\,x + B)\,e^{(5\,x)}$$

From the other hand, $y_{pn}$ is a particular solution of given equation, then

$$y''_{pn} - 4y'_{pn} + 3y_{pn} = (2x+3)e^{5x}\,.$$

Hence,

$$(8Ax + 10A + 8B)e^{5x} = (2x+3)e^{5x}$$

for all $x$. It follows that

$$\begin{cases} 8A = 2 \\ 6A + 8B = 3 \end{cases} \tag{3.49}$$

and we obtain $A = 1/4$, $B = 3/16$.

Thus,

$$y_{pn} = \left(\frac{1}{x}x + \frac{3}{16}\right)e^{5x}\;.$$

The characteristic polynomial $P(k) = k^2 - 4k + 3$ has the roots $k_1 = 1$ and $k_2 = 3$. It follows that the general solution is

$$y = y_{gh} + y_{pn} = C_1 e^x + C_2 e^{3x} + \left(\frac{1}{x}x + \frac{3}{16}\right)e^{5x}\;.$$

**Computations in MAPLE**

```
> ode := diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=(2*x+3)*exp(5*x);
```

$$ode := (\tfrac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) - 4\,(\tfrac{\partial}{\partial x}\,\mathrm{y}(x)) + 3\,\mathrm{y}(x) = (2\,x+3)\,e^{(5\,x)}$$

```
> with(DEtools):
> odeadvisor(ode);
```

$$[[\_2nd\_order,\ \_linear,\ \_nonhomogeneous]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = e^x\,\_C2 + e^{(3\,x)}\,\_C1 + \frac{1}{16}\,(3 + 4\,x)\,e^{(5\,x)}$$

**Example 3.19.** Consider the equation

$$y'' - 4y' + 5y = 7e^{2x}\sin\,x.$$

Here $f(x) = 7e^{5x}$ is a quasi-polynomial of degree $d = 0$ and of the exponent $a \pm bi = 2 \pm i$. Since $a + bi = 2 + i$ (resp., $a - bi = 2 - i$) is a root of multiplicity 1 of the characteristic polynomial

$$P(k) = k^2 - 4k + 5 = (k - (2+i))(k - (2-i))$$

then we have the resonance of order $m = 1$ and there is a particular solution of the form

$$y_{pn} = x \cdot (Ae^{2x}\cos\,x + Be^{2x}\sin\,x)$$

where $A$, $B$ are two coefficients which can be found as in Example 3.18.

**Computations in MAPLE**

```
> ode := diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=7*exp(2*x)*sin(x);
```

$$ode := (\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) - 4\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 5\,\mathrm{y}(x) = 7\,e^{(2\,x)}\sin(x)$$

```
> with(DEtools):
> odeadvisor(ode);
```

$$[[\_2nd\_order,\ \_linear,\ \_nonhomogeneous]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = e^{(2\,x)}\sin(x)\,\_C2 + e^{(2\,x)}\cos(x)\,\_C1 - \frac{7}{2}\,e^{(2\,x)}\cos(x)\,x$$

Thus, we have $A = -7/2$, $B = 0$ and the general solution is

$$y = C_1 e^{2\,x}\,\cos x + C_2 e^{2\,x}\,\sin x - \frac{7}{2}\,x\,e^{2\,x}\,\cos x\,.$$

### 3.9.3. Remarks on linear ODE of higher order with constant coefficients

Linear ODE of higher order with constant coefficients

$$y^{(n)} + c_{n-1}y^{(n-1)} + \ldots + c_1 y' + c_0 y = f(x)\,, \tag{3.50}$$

where $c_i \in \mathbb{R}$ and $f(x)$ is a quasi-polynomial, can be solved in a similar way.

Let us briefly describe the algorithm.

**Step 1.** We need to find the general solution $y_{gh}$ of the homogeneous equation

$$y^{(n)} + c_{n-1}y^{(n-1)} + \ldots + c_1 y' + c_0 y = 0\,. \tag{3.51}$$

To do this we write down the characteristic polynomial

$$P(k) = k^n + c_{n-1}k^{n-1} + \ldots + c_1 k + c_0$$

and then make a factorization of $P(k)$ over $\mathbb{C}$. Suppose that

$$P(k) = \prod_{p=1}^{s}(k-k_p)^{l_p}\cdot\prod_{r=1}^{t}[(k-(a_r+b_r i))(k-(a_r-b_r i))]^{j_r}$$

is the decomposition, the roots $k_1$, $\ldots$, $k_s$ are real of multiplicities $l_1$, $\ldots$, $l_s$ and the roots $a_1 \pm b_1 i$, $\ldots$,$a_t \pm b_t i$ are complex of multiplicities $j_1$, $\ldots$, $j_t$. The the general solution can be written in the form

$$y_{gh} = \sum_{p=1}^{s} F_p(x) + \sum_{r=1}^{t} G_r(x)$$

where

$$F_p(x) = \left(A_{p,1} + A_{p,2}x + \ldots + A_{p,l_p}x^{l_p-1}\right)e^{k_p x}$$

is the general quasi-polynomial of the type I with indefinite coefficients having the degree $d = l_p - 1$ and the exponent $k_p$ and

$$G_r(x) =$$
$$= (B_{r,1} + B_{r,2}x + \ldots + B_{r,j_r}x^{j_r-1})\, e^{a_r x} \cos\, b_r x + (C_{r,1} + C_{r,2}x + \ldots + C_{r,j_r}x^{j_r-1})\, e^{a_r x} \sin\, b_r x$$

is the general quasi-polynomial of the type II with indefinite coefficients having the degree $d = j_r - 1$ and the pair of exponents $\{a_r \pm b_r i\}$.

Note that the total number of coefficients is equal to $n$, that is, the order of equation.

**Example 3.20.** Solve the equation

$$y^{(5)} - 6y^{(4)} + 13y''' - 14y'' + 12y' - 8y = 0\,.$$

**Computations in MAPLE**

```
> ode :=
> diff(y(x),x$5)-6*diff(y(x),x$4)+13*diff(y(x),x$3)-14*diff(y(x),x$2)+12
> *diff(y(x),x)-8*y(x)=0;
```

$$ode := (\tfrac{\partial^5}{\partial x^5}\,\mathrm{y}(x)) - 6\,(\tfrac{\partial^4}{\partial x^4}\,\mathrm{y}(x)) + 13\,(\tfrac{\partial^3}{\partial x^3}\,\mathrm{y}(x)) - 14\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{y}(x)) + 12\,(\tfrac{\partial}{\partial x}\,\mathrm{y}(x)) - 8\,\mathrm{y}(x) = 0$$

```
> P:=k->k^5-6*k^4+13*k^3-14*k^2+12*k-8;  #characteristic polynomial
```

$$P := k \to k^5 - 6\,k^4 + 13\,k^3 - 14\,k^2 + 12\,k - 8$$

```
> factor(P(k));
```

$$(k^2 + 1)\,(k - 2)^3$$

```
> factor(P(k),complex);
```

$$(k + 1.\,I)\,(k - 1.\,I)\,(k - 2.)^3$$

```
> with(DEtools):odeadvisor(ode);
```

$$[[\_high\_order,\ \_missing\_x]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \_C1\, e^{(2\,x)} + \_C2\, e^{(2\,x)}\, x + \_C3\, e^{(2\,x)}\, x^2 + \_C4\, \sin(x) + \_C5\, \cos(x)$$

Thus, the general solution is

$$y = (C_1 + C_2 x + C_3 x^2)\, e^{2\,x} + C_4 \sin x + C_5 \cos x\ .$$

**Step 2.** The following theorem can be proved.

**Theorem 5.**

**1.** If the quasi-polynomial $f(x)$ has the exponent $a$ (or the pair of exponents $\{a \pm bi\}$ for $f(x)$ of the type II) which is not a root of the characteristic polynomial $P(k)$ then there exists a particular solution $y_{pn}$ which is a quasi-polynomial of the same degree and the same exponent as $f(x)$ (no resonance).

**2.** If the quasi-polynomial $f(x)$ has the exponent $a$ (or the pair of exponents $\{a \pm bi\}$ for $f(x)$ of the type II) which is a root of multiplicity $m$ of the characteristic polynomial $P(k)$ then there exists a particular solution $y_{pn}$ which is a quasi-polynomial of the same degree and the same exponent as $f(x)$ multiplied by $x^m$ (resonance of order $m$).

**Example 3.20.** Solve the equation

$$y''' - 3y' + 2y = 54xe^x .$$

Here the characteristic polynomial

$$P(k) = k^3 - 3k + 2 = (k-1)^2(k+2) .$$

The quasi-polynomial $f(x) = 54xe^x$ is of degree $d = 1$ and of exponent $a = 1$. Since $a = 1$ is a root of multiplicity $m = 2$ of $P(k)$ then we have the resonance of second order. It follows from Theorem 5 that there is a particular solution of the equation of the form

$$y_{pn} = x^2 \cdot (Ax + B)e^x .$$

The coefficients $A$, $B$ can be found as above in Example 3.18. We have $A = 3$ and $B = -3$. The general solution is

$$y = y_{gh} + y_{pn} = (C1 + C_2x)e^x + C_3e^{-2x} + x^2 \cdot (-3 + 3x)e^x .$$

**Computations in MAPLE**

```
> ode := diff(y(x),x$3)-3*diff(y(x),x)+2*y(x)=54*x*exp(x);
```

$$ode := (\frac{\partial^3}{\partial x^3}\,\mathrm{y}(x)) - 3\,(\frac{\partial}{\partial x}\,\mathrm{y}(x)) + 2\,\mathrm{y}(x) = 54\,x\,e^x$$

```
> P:=k->k^3-3*k+2;  #characteristic polynomial
```

$$P := k \rightarrow k^3 - 3\,k + 2$$

```
> factor(P(k));
```

$$(k-1)^2\,(k+2)$$

```
> with(DEtools):odeadvisor(ode);
```

$$[[\_3rd\_order,\ \_linear,\ \_nonhomogeneous]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = 3\,x^3\,e^x - 3\,x^2\,e^x + \_C1\,e^x + \_C2\,e^{(-2\,x)} + \_C3\,x\,e^x$$

## 3.10. Incomplete ODE of second order

### 3.10.1. The equation $y'' = f(x)$.

The right-hand part of this equation does not depend on $y$, $y'$. The general solution of this equation can be obtained as follows

$$y' = \int f(x)dx + C_1 ,$$

$$y = \int \left( \int f(x)dx \right) dx + C_1x + C_2 .$$

**Example 3.21.** Solve the equation

$$x = \ln \frac{y''}{x^2} .$$

Let us first rewrite the equation in the equivalent form $y'' = x^2 e^x$ . Then

$$y' = \int x^2 e^x dx = (x^2 - 2x + 2)e^x + C_1 \ ,$$

$$y = \int ((x^2 - 2x + 2)e^x + C_1)dx = (x^2 - 4x + 6)e^x + C_1 x + C_2 \ .$$

**Computations in MAPLE**

```
> ode := diff(y(x),x$2)=x^2*exp(x);
```

$$ode := \frac{\partial^2}{\partial x^2} \mathrm{y}(x) = x^2 \, e^x$$

```
> int(x^2*exp(x),x);
```

$$x^2 \, e^x - 2 \, x \, e^x + 2 \, e^x$$

```
> int(int(x^2*exp(x),x),x);
```

$$x^2 \, e^x - 4 \, x \, e^x + 6 \, e^x$$

```
> with(DEtools):odeadvisor(ode);
```

$$[[\_2nd\_order, \ \_quadrature]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = x^2 \, e^x - 4 \, x \, e^x + 6 \, e^x + \_C1 \, x + \_C2$$

**3.10.2. The equation $y'' = f(x, y')$.**

The right-hand part of this equation does not depend on $y$. An equation of this type can be solved (in quadratures) as follows.

Make the substitution $p = y'$. Then $y'' = f(x, y')$ is equivalent to the system

$$\begin{cases} p' = f(x, p) \\ y' = p \end{cases}$$

The first equation is of first order. Suppose that we can solve this equation (in quadratures). Let $p = P(x, C_1)$ be a general solution where $P$ is known. Then it remains to solve the first order equation

$$y' = P(x, C_1) \ .$$

Integrating we obtain

$$y = \int P(x, C_1)dx + C_2 \ .$$

**Example 3.22.** Solve the equation

$$y'' = 2 + \frac{y'}{x} \ .$$

Following the above method we obtain the system

$$\begin{cases} p' = 2 + \dfrac{p}{x} \\ y' = p \, . \end{cases}$$

The equation

$$p' = 2 + \frac{p}{x}$$

is linear of first order. It has the general solution (say, for $x > 0$) is

$$p = 2x \ln x + C_1 x \ .$$

Further, the equation

$$y' = 2x \ln x + C_1 x$$

has the general solution

$$y = x^2 \ln x - \frac{1}{2}x^2 + \frac{1}{2}C_1 x^2 + C_2 \ .$$

**Computations in MAPLE**

```
> ode1 := diff(p(x),x)=2+p(x)/x;
```

$$ode1 := \frac{\partial}{\partial x}\,\mathrm{p}(x) = 2 + \frac{\mathrm{p}(x)}{x}$$

```
> with(DEtools):odeadvisor(ode1);
```

$$[\_linear]$$

```
> dsolve(ode1);
```

$$\mathrm{p}(x) = (2\ln(x) + \_C1)\,x$$

```
> ode2 :=diff(y(x),x)=(2*ln(x)+_C1)*x;
```

$$ode2 := \frac{\partial}{\partial x}\,\mathrm{y}(x) = (2\ln(x) + \_C1)\,x$$

```
> dsolve(ode2);
```

$$\mathrm{y}(x) = x^2\ln(x) - \frac{1}{2}x^2 + \frac{1}{2}\_C1\,x^2 + \_C2$$

```
> ode:= diff(y(x),x$2)=2+diff(y(x),x)/x;
```

$$ode := \frac{\partial^2}{\partial x^2}\,\mathrm{y}(x) = 2 + \frac{\frac{\partial}{\partial x}\,\mathrm{y}(x)}{x}$$

```
> odeadvisor(ode);
```

$$[[\_2nd\_order, \ \_missing\_y]]$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = x^2\ln(x) - \frac{1}{2}x^2 + \frac{1}{2}\_C1\,x^2 + \_C2$$

**3.10.3. The equation $y'' = f(y, y')$.**

The right-hand part of this equation does not depend on $x$. The equation of this type can be solved as follows.

Make again the substitution $p = y' = dy/dx$. Note that

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p = p'_y \cdot p \ .$$

Then the equation $y'' = f(x, y')$ is equivalent to the system

$$\begin{cases} p'_y = f(y,p)/p \\ y'_x = p \, . \end{cases}$$

The first equation is of first order. Suppose that we can solve this equation (in quadratures). Let $p = P(y, C_1)$ be a general solution where $P$ is known. Then it remains to solve the first order separable (incomplete) equation

$$y' = P(y, C_1)\ .$$

Integrating we obtain the general solution in the form

$$\int \frac{dy}{P(y, C_1)} = x + C_2\ .$$

**Example 3.23.** Solve the equation

$$1 + (y')^2 = 2yy''\ .$$

Make the substitution $p = y' = dy/dx$. Then

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p\ .$$

The original equation can be rewritten in the form

$$1 + p^2 = 2y\,\frac{dp}{dy} \cdot p\,.$$

This is a separable equation, consequently,

$$\frac{2p\,dp}{p^2+1} = \frac{dy}{y}\ ,$$

$$\int \frac{2p\,dp}{p^2+1} = \int \frac{dy}{y}$$

$$\ln(p^2+1) = \ln|y| + \ln C_1\ .$$

It follows that $p^2 + 1 = C_1 y$ and we obtain $p = \pm\sqrt{C_1 y - 1}$.

Recall that $p = dy/dx$. Let us solve the separable equation $\dfrac{dy}{dx} = \pm\sqrt{C_1 y - 1}$ .

$$\pm \int \frac{dy}{\sqrt{C_1 y - 1}} = \int dx\ ,$$

$$\pm\frac{2\sqrt{C_1 y - 1}}{C_1} = x + C_2\ ,$$

$$C_1 y - 1 = \frac{1}{4}\, C_1^2 (x + C_2)^2\ ,$$

and finally we get the general solution

$$y = \frac{4 + C_1^2 (x + C_2)^2}{4C_1}\ .$$

**Computations in MAPLE**

```
> ode:=1+diff(y(x),x)^2=2*y(x)*diff(y(x),x$2);
```

$$ode := 1 + (\tfrac{\partial}{\partial x}\,\mathrm{y}(x))^2 = 2\,\mathrm{y}(x)\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{y}(x))$$

```
> dsolve(ode);
```

$$\mathrm{y}(x) = \frac{1}{4}\,\frac{4 + x^2\,\_C1^2 + 2\,x\,\_C1^2\,\_C2 + \_C2^2\,\_C1^2}{\_C1}$$

## 3.11. Systems of linear equations of first order

In general, an autonomous system of two differential equations of first order on two dependable variables $x$, $y$ ($x = x(t)$, $y = y(t)$) is the following system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y)\,, \end{cases} \tag{3.52}$$

where $f$, $g$ are some known functions ("autonomous" means that these two functions do not depend on $t$), $\dot{x} = dx/dt$, $\dot{y} = dy/dt$.

A solution of (3.52) is a pair $(x(t), y(t))$ (a vector-function) such that the components $x = x(t)$, $y = y(t)$ satisfy (3.52) for each $t$ (from some interval $(a, b)$). The 3-dimensional curve $(x(t), y(t), t)$ , $t \in (a, b)$, is called *integral* curve. The projection of the integral curve on the plane with coordinates $(x, y)$ is called *phase curve* or *phase trajectory.* Thus, a phase curve $K$ can be given in a parametric form as follows

$$K : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

where $(x(t), y(t))$ is a solution of (3.52). The plane with coordinates $(x, y)$ is called *phase plane.*

All phase curves compose a *phase portrait* for a system (3.52).

**Example 3.24.** Consider the second order ODE

$$\ddot{x} = -\sin x$$

(the pendulum equation). If we denote $y = \dot{x}$ then we obtain the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases} \tag{3.53}$$

that in a certain sense is equivalent to the original ODE. Here the coordinates on the phase plane are $x$ (it means the distance from the equilibrium point ) and $y = \dot{x}$ (that is, the velocity). Thus, each point $(x, y)$ corresponds to a certain position and a certain velocity of pendulum. The phase trajectory describes the change of position and velocity when the pendulum oscillates (as the time elapses).

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=y(t),D(y)(t)=-sin(x(t))],
> [x(t),y(t)],
> t=-6..6,[[x(0)=3,y(0)=0.6],[x(0)=0,y(0)=0.8],[x(0)=3.14,y(0)=-0.5],[x(
> 0)=3.14,y(0)=0.3],[x(0)=3.12,y(0)=0],[x(0)=3.15,y(0)=0],[x(0)=0,y(0)=0
> .2],[x(0)=0,y(0)=1.5]],stepsize=.01,
> linecolour=[violet,green,black,red,blue,violet,green,blue],method=clas
> sical[foreuler],color=brown);
```

**Vector fields associated with the system (3.52)**.

Due to (3.52) the tangent vector

$$V = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

to the phase curve

$$K : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

at the point $(x,y)$ is the following vector

$$V = V(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} .$$

The application $(x,y) \to V(x,y)$ defines a vector field on the plane $Oxy$.

**Stationary (equilibrium) points.** Suppose that $(x_0, y_0)$ is a solution of the system of equations

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \,, \end{cases}$$

that is,

$$\begin{cases} f(x_0,y_0) = 0 \\ g(x_0,y_0) = 0 \,. \end{cases}$$

It follows that $(x(t), y(t)) \equiv (x_0, y_0)$ is a solution of (3.52). The corresponding phase curve is just the point $(x_0, y_0)$ on the phase plane. Such a point is called *stationary* or *equilibrium* point for the system (3.52). If a position of a mechanical system $S$ (e.g., a pendulum) is represented by the point $(x_0, y_0)$ on a phase plane then the system $S$ stays in this equilibrium position during infinitely long time.

The stationary point of (3.52) is called *singular* point of the corresponding vector field $V$. At this point the vector $V$ is trivial.

For instance, $(0,0)$ and $(\pi,0)$ are two stationary points for the system (3.53). The first one corresponds to the lowest stable equilibrium position of a pendulum (the pendulum does not oscillate). The second one corresponds to the highest unstable equilibrium position (the pendulum is upset).

**Computations in MAPLE**

```
> with(plots): fieldplot(
> [y,-sin(x)],x=-4..7,y=-3..3,color=violet,arrows=slim,grid=[20,20]);
```

**Linear systems**

In this subsection we consider only the linear systems

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \end{cases} \tag{3.54}$$

where $x = x(t)$, $y = y(t)$ are two unknown functions,

A vector-function also can be written as a column

$$X = X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

If we consider the matrix of coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then (3.54) can be written in the matrix form as follows

$$\dot{X} = A \cdot X \,. \tag{3.55}$$

Assume that $A$ is non-degenerate matrix. Then there is the unique equilibrium point, namely, $(0, 0)$ for the system (3.54) since the linear system

$$\begin{cases} a_{11}x + a_{12}y = 0 \\ a_{21}x + a_{22}y = 0 \end{cases}$$

has only trivial solution.

**Classification of stationary points for planar linear systems.**

The stationary points for planar linear systems $\dot{X} = A \cdot X$ with a non-degenerate matrix $A$ can be classified. There are several types (up to linear change of variables). It turns out that the type of the stationary point depends on the eigenvalues $\lambda_1$, $\lambda_2$ of the matrix $A$.

| **Type** | **Eigenvalues of $A$** | | **Example** |
|---|---|---|---|
| Unstable node | $0 < \lambda_1 < \lambda_2\,;$ | $\lambda_{1,2} \in \mathbb{R}$ | $\dot{x} = 2y$<br>$\dot{y} = -x + 3y$ |
| Degenerate unstable node | $0 < \lambda_1 = \lambda_2\,;$ $^{(*)}$ | $\lambda_{1,2} \in \mathbb{R}$ | $\dot{x} = y$<br>$\dot{y} = -x + 2y$ |
| Stable node | $\lambda_1 < \lambda_2 < 0\,;$ | $\lambda_{1,2} \in \mathbb{R}$ | $\dot{x} = -3x + 2y$<br>$\dot{y} = -x$ |
| Degenerate stable node | $\lambda_1 = \lambda_2 < 0\,;$ $^{(*)}$ | $\lambda_{1,2} \in \mathbb{R}$ | $\dot{x} = x - 4y$<br>$\dot{y} = x - 3y$ |
| Saddle | $\lambda_1 < 0 < \lambda_2\,;$ | $\lambda_{1,2} \in \mathbb{R}$ | $\dot{x} = -x - 3y$<br>$\dot{y} = y$ |
| Unstable focus | $\lambda_{1,2} = a \pm bi,\ a > 0;$ | $\lambda_{1,2} \in \mathbb{C}$ | $\dot{x} = x + 13y$<br>$\dot{y} = -2x + y$ |
| Stable focus | $\lambda_{1,2} = a \pm bi,\ a < 0;$ | $\lambda_{1,2} \in \mathbb{C}$ | $\dot{x} = -x + 13y$<br>$\dot{y} = -2x - y$ |
| Center | $\lambda_{1,2} = \pm bi\,;$ | $\lambda_{1,2} \in \mathbb{C}$ | $\dot{x} = x - 2y$<br>$\dot{y} = 3x - y$ |

$(*)$ *the matrix $A$ is assumed not to have a diagonal form.*

In what follows the classification type and the typical phase portrait are presented.

**1. Example of unstable node.** The stationary point of the system

$$\begin{cases} \dot{x} = 2y \\ \dot{y} = -x + 3y \end{cases}$$

is an unstable node.

**Computations in MAPLE.**

```
> with(DEtools):
> phaseportrait([D(x)(t)=2*y(t),D(y)(t)=-x(t)+3*y(t)],
> [x(t),y(t)],
> t=-2..0.1,[[x(0)=0,y(0)=1],[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=1],[x(0)=1,
> y(0)=-1],[x(0)=0,y(0)=-0.8],[x(0)=-1,y(0)=-0.7],[x(0)=0.8,y(0)=0],[x(0
> )=1,y(0)=0.7],[x(0)=1,y(0)=0.5],[x(0)=-0.5,y(0)=0.5],[x(0)=-1,y(0)=-0.
> 8],[x(0)=-1,y(0)=-1],[x(0)=-1,y(0)=-0.6]],stepsize=.01,linecolor=[gree
> n,violet,green,violet,red,black,red,black,blue,blue,violet,blue,red],m
> ethod=classical[foreuler],color=brown);
```

**2. Example of degenerate unstable node.** The equilibrium point of the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + 2y \end{cases}$$

is a degenerate unstable node.

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=y(t),D(y)(t)=-x(t)+2*y(t)],
> [x(t),y(t)],
> t=-4..0.2,[[x(0)=0,y(0)=1],[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=1],[x(0)=1,
> y(0)=-1],[x(0)=0,y(0)=-0.8],[x(0)=-1,y(0)=-0.7],[x(0)=0.7,y(0)=-0.3],[
> x(0)=1,y(0)=0.7],[x(0)=1,y(0)=0.5],[x(0)=-0.5,y(0)=0.5],[x(0)=-0.7,y(0
> )=0.2],[x(0)=-1,y(0)=-0.4],[x(0)=-1,y(0)=-0.6],[x(0)=-1,y(0)=-1]],step
> size=.01,linecolor=[green,violet,green,violet,red,black,red,black,blue
> ,blue,violet,blue,red,violet],method=classical[foreuler],color=brown);
```

**3. Example of stable node.** The stationary point of the system

$$\begin{cases} \dot{x} = -3x + 2y \\ \dot{y} = -x \end{cases}$$

is a stable node.

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=-3*x(t)+2*y(t),D(y)(t)=-x(t)],
> [x(t),y(t)],
> t=-0.2..5,[[x(0)=0,y(0)=1],[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=1],[x(0)=1,
> y(0)=-1],[x(0)=0,y(0)=-0.8],[x(0)=-1,y(0)=-0.7],[x(0)=0.8,y(0)=0],[x(0
> )=1,y(0)=0.7],[x(0)=1,y(0)=0.5],[x(0)=-0.5,y(0)=0.5],[x(0)=-1,y(0)=-0.
> 8],[x(0)=-1,y(0)=-1],[x(0)=-1,y(0)=-0.6]],stepsize=.01,linecolor=[gree
> n,violet,green,violet,red,black,red,black,blue,blue,violet,blue,red],m
> ethod=classical[foreuler],color=brown);
```

**4. Example of degenerate stable node.** The stationary point of the system

$$\begin{cases} \dot{x} = x - 4y \\ \dot{y} = x - 3y \end{cases}$$

is a degenerate stable node.

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=x(t)-4*y(t),D(y)(t)=x(t)-3*y(t)],
> [x(t),y(t)],
> t=-0.2..5,[[x(0)=0,y(0)=1],[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=1],[x(0)=1,
> y(0)=-1],[x(0)=0,y(0)=-0.8],[x(0)=-1,y(0)=-0.7],[x(0)=0.8,y(0)=0],[x(0
> )=1,y(0)=0.7],[x(0)=1,y(0)=0.5],[x(0)=-0.5,y(0)=0.5],[x(0)=-1,y(0)=-0.
> 8],[x(0)=-1,y(0)=-1],[x(0)=-1,y(0)=-0.6]],stepsize=.01,linecolor=[gree
> n,violet,green,violet,red,black,red,black,blue,blue,violet,blue,red],m
> ethod=classical[foreuler],color=brown);
```

**5. Example of saddle.** The stationary point of the system

$$\begin{cases} \dot{x} = -x - 3y \\ \dot{y} = y \end{cases}$$

is a saddle.

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=-x(t)-3*y(t),D(y)(t)=y(t)],
> [x(t),y(t)],
> t=-1...1.1,[[x(0)=0.3,y(0)=0.7],[x(0)=0,y(0)=-0.5],[x(0)=0,y(0)=0.5],[
> x(0)=-1.6,y(0)=0.6],[x(0)=-1,y(0)=0.3],[x(0)=1.2,y(0)=-0.3],[x(0)=-0.3
> ,y(0)=-0.7],[x(0)=2,y(0)=-0.5],[x(0)=-0.6,y(0)=0.4],[x(0)=0.6,y(0)=-0.
> 4],[x(0)=-0.8,y(0)=0],[x(0)=0.8,y(0)=0]],stepsize=.01,linecolor=[green
> ,violet,green,violet,red,black,red,black,blue,blue,red,violet],method=
> classical[foreuler],color=brown);
```

**6. Example of unstable focus.** The stationary point of the system

$$\begin{cases} \dot{x} = x + 13y \\ \dot{y} = -2x + y \end{cases}$$

is an unstable focus.

**Computations in MAPLE**

```
> with(DEtools):

> phaseportrait([D(x)(t)=x(t)+13*y(t),D(y)(t)=-2*x(t)+y(t)],
> [x(t),y(t)],
> t=-3..1,[[x(0)=0.1,y(0)=0.1],[x(0)=-0.1,y(0)=0.1],[x(0)=-0.1,y(0)=-0.1
> ],[x(0)=0.1,y(0)=-0.1]],stepsize=.01,linecolor=[green,violet,blue,red]
> ,method=classical[foreuler],color=brown);
```

**7. Example of stable focus.** The stationary point of the system

$$\begin{cases} \dot{x} = -x + 13y \\ \dot{y} = -2x - y \end{cases}$$

is a stable focus.

**Computations in MAPLE**

```
> with(DEtools):

> phaseportrait([D(x)(t)=-x(t)+13*y(t),D(y)(t)=-2*x(t)-y(t)],
> [x(t),y(t)],
> t=-1..3,[[x(0)=0.1,y(0)=0.1],[x(0)=-0.1,y(0)=0.1],[x(0)=-0.1,y(0)=-0.1
> ],[x(0)=0.1,y(0)=-0.1]],stepsize=.01,linecolor=[green,violet,blue,red]
> ,method=classical[foreuler],color=brown);
```

**8. Example of center.** The stationary point of the system

$$\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 3x - y \end{cases}$$

is a center.

**Computations in MAPLE**

```
> with(DEtools):
> phaseportrait([D(x)(t)=x(t)-2*y(t),D(y)(t)=3*x(t)-y(t)],
> [x(t),y(t)],
> t=-1.5..1.5,[[x(0)=0.3,y(0)=0.3],[x(0)=0.6,y(0)=0.6],[x(0)=1,y(0)=1],[
> x(0)=1.5,y(0)=1.5],[x(0)=0.1,y(0)=0.1]],stepsize=.005,linecolor=[green
> ,violet,red,blue,black],method=classical[foreuler],color=brown);
```

**Solutions of linear systems.**

Let suppose that the matrix $A$ of a linear system

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \end{cases}$$

has two distinct eigenvalues $\lambda_1 \neq \lambda_2$. One can prove the following result.

**I.** If $\lambda_1$, $\lambda_2$ are real then each function $x(t)$, $y(t)$ of a solution $(x(t), y(t))$ is a linear combination of functions $e^{\lambda_1 t}$, $e^{\lambda_2 t}$.

**II.** If $\lambda_{1,2} = a \pm bi$ are complex then each function $x(t)$, $y(t)$ of a solution $(x(t), y(t))$ is a linear combination of functions $e^{at}\cos bt$, $e^{at}\sin bt$.

Practically, the coefficients of linear combinations can be found by the indefinite coefficients method (see Example 3.18).

In general, the initial condition problem for the linear system

$$\dot{X} = AX\,, \quad X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

has the solution

$$X = \exp(tA) \cdot X_0\,. \tag{3.56}$$

**Example 3.25.** Solve the initial condition problem

$$\begin{cases} \dot{x} = x + 12y \\ \dot{y} = 3x + y\,, \end{cases}$$

$$x(0) = 1\,,\ y(0) = 1\,.$$

The eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$$

are $\lambda_1 = -5$ and $\lambda_2 = 7$ (thus, the stationary point $(0,0)$ is a saddle point). First, let us find the general solution of the system. It follows from the above result that $x = x(t)$ can be written in the form

$$x = C_1 e^{-5t} + C_2 e^{7t}\,,$$

hence

$$\dot{x} = -5\,C_1 e^{-5t} + 7\,C_2 e^{7t}\,.$$

We obtain by the first equation of the system that

$$y = \frac{1}{12}\,(\dot{x} - x) = -\frac{1}{2}\,C_1 e^{-5t} + \frac{1}{2}\,C_2 e^{7t}\,.$$

Consequently, the general solution is

$$x = C_1 e^{-5t} + C_2 e^{7t}\,, \quad y = -\frac{1}{2}\,C_1 e^{-5t} + \frac{1}{2}\,C_2 e^{7t} \tag{3.57}$$

where $C_1$, $C_2$ are two arbitrary constants.

Let us now solve the initial condition problem. Substituting $t = 0$ into (3.57) we get

$$1 = x(0) = C_1 + C_2\,, \quad 1 = y(0) = -\frac{1}{2}\,C_1 + \frac{1}{2}\,C_2\,.$$

It follows that $C_1 = -\frac{1}{2}$, $C_2 = \frac{3}{2}$. Hence

$$x = -\frac{1}{2}\,e^{-5t} + \frac{3}{2}\,e^{7t}\,, \quad y = \frac{1}{4}\,e^{-5t} + \frac{3}{4}\,e^{7t}$$

is the solution of the problem.

**Computations in MAPLE**

```
> restart;
> sys:= {diff(x(t),t) = x(t)+12*y(t), diff(y(t),t) = 3*x(t)+y(t)};
```

$$sys := \{\tfrac{\partial}{\partial t}\,\mathrm{x}(t) = \mathrm{x}(t) + 12\,\mathrm{y}(t),\ \tfrac{\partial}{\partial t}\,\mathrm{y}(t) = 3\,\mathrm{x}(t) + \mathrm{y}(t)\}$$

```
> IC := {x(0)=1,y(0)=1};   #Initial conditions
```

$$IC := \{\mathrm{x}(0) = 1,\ \mathrm{y}(0) = 1\}$$

```
> dsolve(sys);
```

$$\{\mathrm{x}(t) = \_C1\,e^{(-5\,t)} + \_C2\,e^{(7\,t)},\ \mathrm{y}(t) = -\frac{1}{2}\,\_C1\,e^{(-5\,t)} + \frac{1}{2}\,\_C2\,e^{(7\,t)}\}$$

```
> dsolve(sys union IC);
```

$$\{\mathrm{y}(t) = \frac{1}{4}\,e^{(-5\,t)} + \frac{3}{4}\,e^{(7\,t)},\ \mathrm{x}(t) = -\frac{1}{2}\,e^{(-5\,t)} + \frac{3}{2}\,e^{(7\,t)}\}$$

```
> with(LinearAlgebra):
> A:=Matrix([[1,12],[3,1]]);     #Matrix of the system
```

$$A := \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix}$$

```
> lambda:=Eigenvalues(A);
```

$$\lambda := \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

Let us check the formula (3.56) for this example.

**Computations in MAPLE**

```
> with(LinearAlgebra):
> with(linalg):
> A:=Matrix([[1,12],[3,1]]);
```

$$A := \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix}$$

```
> X0:=Vector([1,1]);
```

$$X0 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
> V:=exponential(t*A);
```

$$V := \begin{bmatrix} \frac{1}{2}\,e^{(-5\,t)} + \frac{1}{2}\,e^{(7\,t)} & e^{(7\,t)} - e^{(-5\,t)} \\ \frac{1}{4}\,e^{(7\,t)} - \frac{1}{4}\,e^{(-5\,t)} & \frac{1}{2}\,e^{(-5\,t)} + \frac{1}{2}\,e^{(7\,t)} \end{bmatrix}$$

```
> f:= (i,j) -> V[i,j]:  W:=Matrix(2,f);
```

$$\mathrm{W} := \begin{bmatrix} \frac{1}{2}\,e^{(-5\,t)} + \frac{1}{2}\,e^{(7\,t)} & e^{(7\,t)} - e^{(-5\,t)} \\ \frac{1}{4}\,e^{(7\,t)} - \frac{1}{4}\,e^{(-5\,t)} & \frac{1}{2}\,e^{(-5\,t)} + \frac{1}{2}\,e^{(7\,t)} \end{bmatrix}$$

```
> Answer:=W.X0;
```

$$Answer := \begin{bmatrix} -\frac{1}{2}\,e^{(-5\,t)} + \frac{3}{2}\,e^{(7\,t)} \\ \frac{3}{4}\,e^{(7\,t)} + \frac{1}{4}\,e^{(-5\,t)} \end{bmatrix}$$