4. Elements of calculus of variations

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This section is devoted to three basic (1-dimensional) problems of classical calculus of variations.

4.1. Classical variational problem

Historical remarks. The problem of brachistochrone.

In 1696 J. Bernoulli stated the following problem.

To find the curve joining two given points along which a particle accelerated by the gravity falls for the least time.

Such a curve is called *brachistochrone*. In the modern mathematical language this problem can be stated as follows. Suppose, for instance, that the starting point is the point $A = (0, 0)$ on the coordinate plane Oxy and the end point is $B = (x_1, y_1)$. The axis Oy is directed downwards (this is the direction of gravity acceleration vector). Assume that the curve K (brachistochrone) joining the points A, B is given as the plot of a function $y = y(x)$.

We may write down the expression for the elapsed time T of motion along K using curvilinear integral

$$T = \int_K \frac{ds}{v}$$

where v is the absolute value of velocity.

The law of energy conservation implies the relation $v^2 = 2gy$ (here $g = 9.8\,m/s^2$ is the acceleration of gravity) which holds at each moment. In the coordinates x, y the curvilinear integral can be transformed as below ($ds = \sqrt{1 + (y')^2}\,dx$). Consequently, we obtain the following optimization problem: to find a function $y = y(x)$ which minimizes the functional of time:

$$T = \int_0^{x_0} \frac{\sqrt{1 + (y')^2}\,dx}{\sqrt{2gy}} \to \min ,$$

$$y(0) = 0 \ , \ y(x_1) = y_1 \ .$$

Remark. It turned out that the brachistochrone is the same curve as *cycloid* and can be given parametrically as follows

$$K : \begin{cases} x = C(\tau - \sin\tau) + a \\ y = C(1 - \cos\tau) \end{cases}$$

for appropriate constants C, a.

The problem of brachistochrone is considered now as the origin of calculus of variations. The classical general methods for solving variational problems were elaborated by L.Euler and J.L. Lagrange in the 18th century. A close relationship between calculus of variations and mechanics was established. Now this theory concerns a big circle of problems.

Statement of the classical problem *Pr1.* Starting with the problem of brachistochrone we can state the following extremal problem which is called *the classical problem of calculus of variations.*

In the space $C^{(1)}[t_0,\, t_1]$ of all continuously differentiable functions $x = x(t)$, $t \in [t_0,\, t_1]$, to find all functions such that

1. *function $x = x(t)$ minimizes or maximizes the value of the functional $\mathcal{J}$:*

$$\mathcal{J}[x(t)] = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \;\to \text{extr}$$

   where L is a given smooth function (of class $C^{(1)}$) on three variables, and

2. *$x(t_0) = x_0,\ x(t_1) = x_1$ (boundary conditions).*

We will refer to this problem as *Pr1.* In this problem the function L is called *integrant.* The functions $x(t) \in C^{(1)}[t_0,\, t_1]$ which satisfy the boundary conditions are called *admissible.*

Definition 1. We say that the admissible function $\hat{x}(t)$ provides a (weak) local minimum to $\mathcal{J}$ in the problem *Pr1* if the inequality

$$\mathcal{J}[\hat{x}(t)] \leqslant \mathcal{J}[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we will write

$$\hat{x}(t) \in \text{locmin } \textit{Pr1}\ .$$

Definition 2. We say that the admissible function $\hat{x}(t)$ provides a (weak) local maximum to $\mathcal{J}$ in the problem *Pr1* if the inequality

$$\mathcal{J}[\hat{x}(t)] \geqslant \mathcal{J}[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we will write

$$\hat{x}(t) \in \text{locmax } \textit{Pr1}\ .$$

In both cases such a function $\hat{x}(t)$ is called *an extremal* or *stationary function* for the functional $\mathcal{J}$ of the problem.

Let us suppose that $\hat{x}(t)$ provides the absolute minimum or absolute maximum to $\mathcal{J}$ in the above problem ($\hat{x}(t) \in$ absmin *Pr1* or $\hat{x}(t) \in$ absmax *Pr1*) . Evidently, $\hat{x}(t)$ provides also a local extremum, consequently, $\hat{x}(t)$ should be an extremal of $\mathcal{J}$. Thus, we should look for the solution of the problem among all extremals of $\mathcal{J}$.

Necessary conditions for extremals in the classical variational problem.

Consider the integrant $L = L(t, x, \dot{x})$. Given a function $\hat{x}(t)$ we will use the following notation:

$$\hat{L} = L(t, \hat{x}, \dot{\hat{x}})\ , \quad \hat{L}_x = \frac{\partial L}{\partial x}(t, \hat{x}, \dot{\hat{x}})\ , \quad \hat{L}_{\dot{x}} = \frac{\partial L}{\partial \dot{x}}(t, \hat{x}, \dot{\hat{x}})$$

(making the partial differentiation we assume that the arguments in $L(t, x, \dot{x})$ are independent variables).

One can prove the following theorem which provides necessary conditions for a function $\hat{x}(t)$ to be an extremal for $\mathcal{J}$.

Theorem 1. Let $\hat{x}(t)$ provide a (weak) local extremum in *Pr1.* Suppose that the integrant L is a function of class $C^{(1)}$ in some neighborhood of the set $\{(t, \hat{x}, \dot{\hat{x}}) | \, t \in [t_0, t_1]\}$. Then the Euler-Lagrange equation

$$\frac{d}{dt}\hat{L}_{\dot{x}} = \hat{L}_x \tag{4.1}$$

is satisfied.

It follows that for finding extremals we need to solve the boundary value problem for the differential equation (4.1) of second order since for any extremal of $\mathcal{J}$

$$\frac{d}{dt}\hat{L}_{\dot{x}} = \hat{L}_x \, ; \qquad \hat{x}(t_0) = x_0 \, , \; \hat{x}(t_1) = x_1 \, . \tag{4.2}$$

Sketch of the proof. Variation of the functional.

Given an arbitrary fixed function $h(x) \in C^{(1)}[t_0, \, t_1]$ such that $h(t_0) = h(t_1) = 0$ and $\lambda \in \mathbb{R}$ consider the following function

$$\Phi(\lambda) = \mathcal{J}[\hat{x}(t) + \lambda \, h(t)] = \int_{t_0}^{t_1} L(t, \hat{x}(t) + \lambda \, h(t), \dot{\hat{x}}(t) + \lambda \, \dot{h}(t)) dt \, .$$

Since $\hat{x}(t) \in \operatorname{locextr} Pr1$ then $\Phi(\lambda)$ has a local extremum at $\lambda = 0$. Consequently, $\Phi'(0) = 0$ (the conditions of Theorem 1 provide the existence of $\Phi'(0)$).

By definition,

$$\delta\mathcal{J}\,(\hat{x}(t))[h(t)] = \lim_{\lambda \to 0} \frac{\mathcal{J}[\hat{x}(t) + \lambda \, h(t)] - \mathcal{J}[\hat{x}(t)]}{\lambda} = \Phi'(0)$$

is called *variation* of $\mathcal{J}$ (applied to h).

It follows that

$$\Phi'(0) = \delta\mathcal{J}\,(\hat{x}(t))[h(t)] = 0$$

if $\hat{x}(t) \in \operatorname{locextr} Pr1$, that is, the variation is trivial for any function $h(x) \in C^{(1)}[t_0, \, t_1]$.

Let us compute $\delta\mathcal{J}\,(\hat{x}(t))[h(t)]$. Differentiating the integral depending on parameter λ we obtain

$$0 = \Phi'(0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \int_{t_0}^{t_1} \Big(L(t, \hat{x}(t) + \lambda \, h(t), \dot{\hat{x}}(t) + \lambda \, \dot{h}(t)) - L(t, \hat{x}(t), \dot{\hat{x}}(t)) \Big) dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{d}{d\lambda} L(t, \hat{x}(t) + \lambda \, h(t), \dot{\hat{x}}(t) + \lambda \, \dot{h}(t)) \right)\Bigg|_{\lambda=0} dt = \int_{t_0}^{t_1} \Big(\hat{L}_x \, h(t) + \hat{L}_{\dot{x}} \, \dot{h}(t) \Big) dt \, .$$

Note that

$$\int_{t_0}^{t_1} \hat{L}_{\dot{x}} \, \dot{h}(t) \, dt = \hat{L}_{\dot{x}} \, h(t) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\hat{L}_{\dot{x}} \, h(t) \, dt = - \int_{t_0}^{t_1} \frac{d}{dt}\hat{L}_{\dot{x}} \, h(t) \, dt$$

since $h(t_0) = h(t_1) = 0$.

Finally, we obtain

$$\Phi'(0) = \delta\mathcal{J}\,(\hat{x}(t))[h(t)] = \int_{t_0}^{t_1} \left(\hat{L}_x - \frac{d}{dt}\hat{L}_{\dot{x}} \right)\, h(t)\,dt = 0\,.$$

The last equality is valid for each function $h(t)$ such that $h(t_0) = h(t_1) = 0$. Consequently (by Dubois-Raymond Lemma),

$$\hat{L}_x - \frac{d}{dt}\hat{L}_{\dot{x}} = 0$$

and we obtain the Euler-Lagrange equation.

Example 4.1. Consider the following example. We need to find all continuously differentiable functions $x = x(t)$ satisfying the conditions $x(0) = 1$, $x(1) = 3$ which minimize the value of the functional

$$\mathcal{J}[x(t)] = \int_0^1 \dot{x}(t)^2\,dt\,.$$

On Figure 1 you can see three such functions.

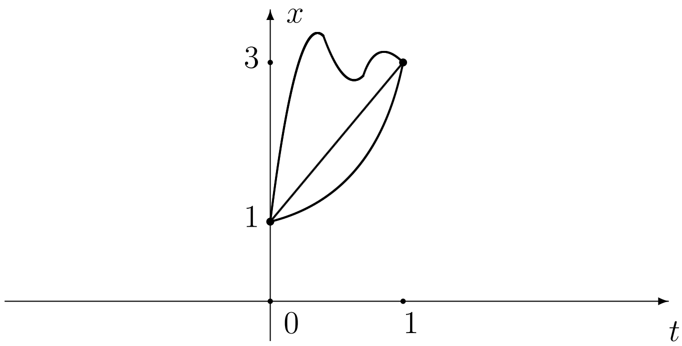

Fig. 1

In this example we have

$$L = \dot{x}^2\,, \quad \frac{\partial L}{\partial x} = 0\,, \quad \frac{\partial L}{\partial \dot{x}} = 2\dot{x}\,.$$

The boundary value problem (4.2) can be stated as follows

$$\frac{d}{dt}(2\dot{x}) = 2\ddot{x} = 0\,; \quad x(0) = 1\,,\, x(1) = 3\,.$$

Solving the ODE $\ddot{x} = 0$ we obtain the extremal of the form $x = C_1 t + C_2$. The boundary conditions provide

$$1 = x(0) = C_2\,, \qquad 3 = x(1) = C_1 + C_2\,.$$

It follows that $C_1 = 2$. Consequently, the unique extremal is

$$\hat{x} = 2t + 1$$

(a straight line on Figure 1).

One can show that $\hat{x} \in$ absmin Pr, that is, $\hat{x}$ provides the absolute minimum to $\mathcal{J}$, $\mathcal{J}_{\min} = 4$. In fact, for each function $h(x) \in C^{(1)}[0,\,1]$ such that $h(0) = h(1) = 0$ we have

$$\int_0^1 \dot{h}(t)\,dt = h(1) - h(0) = 0\,.$$

Then

$$\mathcal{J}[\hat{x}(t)+\ h(t)]-\mathcal{J}[\hat{x}(t)]=\int_0^1\Big((\dot{\hat{x}}(t)+\dot{h}(t))^2-\dot{\hat{x}}(t)^2\Big)\,dt=\int_0^1\Big(2\dot{\hat{x}}(t)\dot{h}(t)+\dot{h}(t)^2\Big)\,dt=$$

$$=\int_0^1\Big(4\dot{h}(t)+\dot{h}(t)^2\Big)\,dt=\int_0^1 4\dot{h}(t)\,dt+\int_0^1\dot{h}(t)^2\,dt=\int_0^1\dot{h}(t)^2\,dt\geqslant 0\,.$$

Consequently,

$$\mathcal{J}[x(t)]=\mathcal{J}[\hat{x}(t)+\ h(t)]\geqslant\mathcal{J}[\hat{x}(t)]=\int_0^1 4\,dt=4$$

for any admissible function $x(t)$.

Computations in MAPLE.

```
> restart;
> ode:= diff(x(t),t$2) = 0;
```

$$ode:=\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)=0$$

```
> BC :=       x(0)=1,x(1)=3;   #Boundary conditions
```

$$BC:=\mathrm{x}(0)=1,\,\mathrm{x}(1)=3$$

```
> dsolve({ode,BC} );
```

$$\mathrm{x}(t)=2\,t+1$$

```
> J:=int(diff((2*t+1),t)^2,t=0..1);
```

$$J:=4$$

Example 4.2. Solve the following problem.

$$\mathcal{J}[x(t)]=\int_0^1(x^2-\dot{x}^2)\,e^{2t}\,dt\ \to\text{extr}\ ,$$

$$x(0)=0\ ,\ x(1)=e\ .$$

Here $L=(x^2-\dot{x}^2)\,e^{2t}$. Consequently,

$$\frac{\partial L}{\partial x}=2x\,e^{2t}\ ,\quad \frac{\partial L}{\partial\dot{x}}=-2\dot{x}\,e^{2t}\ .$$

The boundary value problem for the Euler-Lagrange equation can be stated as follows

$$\frac{d}{dt}(-2\dot{x}\,e^{2t})=2x\,e^{2t}\,;\quad x(0)=0\ ,\ x(1)=e\,,$$

that is,

$$\frac{d}{dt}(-2\dot{x}\,e^{2t})=2x\,e^{2t}\,;\quad x(0)=0\ ,\ x(1)=e\,.$$

Differentiating and reducing by e^{2t} we finally obtain the linear ODE of second order with constant coefficients

$$\ddot{x}+2\dot{x}+x=0\,;\quad x(0)=0\ ,\ x(1)=e\,.$$

The characteristic polynomial $P(k) = k^2 + 2k + 1 = (k+1)^2$ has the unique root $k_1 = -1$ of multiplicity 2. It follows that the extremals are of the form

$$x(t) = (C_1 + C_2 t)\, e^{-t} \ .$$

The boundary conditions imply

$$0 = x(0) = C_1 \ , \quad e = x(1) = (C_1 + C_2)\, e^{-1} \ .$$

It follows that

$$C_1 = 0 \ , \quad C_2 = e^2 \ .$$

Consequently,

$$\hat{x}(t) = e^2\, t\, e^{-t} = t\, e^{2-t}$$

is the unique extremal for the functional $\mathcal{J}$. It can be proved that $\hat{x}(t) \in \text{absmax}\ Pr$. Thus,

$$\mathcal{J}_{\max} = \mathcal{J}[\hat{x}(t)] = 0 \ .$$

Computations in MAPLE.

```
> restart;
> ode:= diff(x(t),t$2)+2*diff(x(t),t)+x(t) = 0;
```

$$ode := (\frac{\partial^2}{\partial t^2}\,\mathrm{x}(t)) + 2\,(\frac{\partial}{\partial t}\,\mathrm{x}(t)) + \mathrm{x}(t) = 0$$

```
> BC :=       x(0)=0,x(1)=exp(1);   #Boundary conditions
```

$$BC := \mathrm{x}(0) = 0,\, \mathrm{x}(1) = e$$

```
> dsolve(ode );
```

$$\mathrm{x}(t) = \_C1\; e^{(-t)} + \_C2\; e^{(-t)}\, t$$

```
> dsolve({ode,BC} );simplify(%);
```

$$\mathrm{x}(t) = \frac{e\, e^{(-t)}\, t}{e^{(-1)}}$$

$$\mathrm{x}(t) = t\, e^{(2-t)}$$

```
> J:=Int((diff(t*exp(2-t),t)^2-(t*exp(2-t))^2)*exp(2*t),t=0..1);
> #symbolic form
```

$$J := \int_0^1 ((e^{(2-t)} - t\, e^{(2-t)})^2 - t^2\, (e^{(2-t)})^2)\, e^{(2\,t)}\, dt$$

```
> J:=int((diff(t*exp(2-t),t)^2-(t*exp(2-t))^2)*exp(2*t),t=0..1);
```

$$J := 0$$

4.2. Bolza problem

Statement of the Bolza problem *Pr2.*

In the space $C^{(1)}[t_0,\, t_1]$ *of all continuously differentiable functions* $x = x(t)$, $t \in [t_0,\, t_1]$, *to find the functions such that function* $x = x(t)$ *minimizes or maximizes the value of the functional* $\mathcal{B}$*:*

$$\mathcal{B}[x(t)] = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt + l(x(t_0), x(t_1)) \;\to \text{extr}$$

where L *is a given function (of class* $C^{(1)}$*) on three variables and* l *is a given function (of class* $C^{(1)}$*) on two variables.*

We will refer to this problem as *Pr2*. In this problem the function L is called *integrant*, the function l is called *terminant*. All functions $x(t) \in C^{(1)}[t_0,\, t_1]$ are admissible, there is no boundary constraints.

Definition 3. We say that the admissible function $\hat{x}(t)$ provides a (weak) local minimum to $\mathcal{B}$ in the problem *Pr2* if the inequality

$$\mathcal{B}[\hat{x}(t)] \leqslant \mathcal{B}[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we write

$$\hat{x}(t) \in \text{locmin } \mathit{Pr2}\,.$$

Definition 4. We say that the admissible function $\hat{x}(t)$ provides a (weak) local maximum to $\mathcal{B}$ in the problem *Pr2* if the inequality

$$\mathcal{B}[\hat{x}(t)] \geqslant \mathcal{B}[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we will write

$$\hat{x}(t) \in \text{locmax } \mathit{Pr2}\,.$$

The other notions of *Pr1* remain valid for this problem.

Necessary conditions for extremals in the Bolza problem.

Consider the integrant $L = L(t, x, \dot{x})$. Given a function $\hat{x}(t)$ we will use the previous notation:

$$\hat{L} = L(t, \hat{x}, \dot{\hat{x}})\ , \quad \hat{L}_x = \frac{\partial L}{\partial x}(t, \hat{x}, \dot{\hat{x}})\ , \quad \hat{L}_{\dot{x}} = \frac{\partial L}{\partial \dot{x}}(t, \hat{x}, \dot{\hat{x}})$$

and the notation

$$\hat{l}_{x_0} = \frac{\partial l}{\partial x_0}(\hat{x}(t_0), \hat{x}(t_1))\ , \quad \hat{l}_{x_1} = \frac{\partial l}{\partial x_1}(\hat{x}(t_0), \hat{x}(t_1))$$

for the terminant l (recall that making the partial differentiation we assume that the arguments in $L(t, x, \dot{x})$ or in $l(x(t_0), x(t_1))$ are independent variables).

One can prove the following theorem which provides necessary conditions for a function $\hat{x}(t)$ to be an extremal for $\mathcal{B}$.

Theorem 2. Let $\hat{x}(t)$ provide a (weak) local extremum in *Pr2*. Suppose that the integrant L is a function of class $C^{(1)}$ in some neighborhood of the set $\{(t, \hat{x}, \dot{\hat{x}}) |\, t \in [t_0, t_1]\}$ and that the terminant l is a function of class $C^{(1)}$ in some neighborhood of the point $(\hat{x}(t_0), \hat{x}(t_1))$. Then

1) the Euler-Lagrange equation

$$\frac{d}{dt}\hat{L}_{\dot{x}} = \hat{L}_x$$

and

2) the transversality conditions

$$\hat{L}_{\dot{x}}(t_0) = \hat{l}_{x_0} \ , \quad \hat{L}_{\dot{x}}(t_1) = -\hat{l}_{x_1} \tag{4.3}$$

are satisfied.

It follows again that for finding extremals we need to solve the boundary value problem for the differential equation (4.1).

Example 4.3. Solve the following problem.

$$\mathcal{B}[x(t)] = \int_0^1 (\dot{x}^2 - x)\, dt + x(1)^2 \ \rightarrow \text{extr} \ ,$$

Here $L = \dot{x}^2 - x$, $l = x(1)^2$. Consequently,

$$\frac{\partial L}{\partial x} = -1 \ , \quad \frac{\partial L}{\partial \dot{x}} = 2\dot{x} \qquad \frac{\partial l}{\partial x(0)} = 0 \qquad \frac{\partial l}{\partial x(1)} = 2x(1) \ .$$

The boundary value problem for the Euler-Lagrange equation can be stated as follows

$$\frac{d}{dt}(2\dot{x}) = -1 \, ; \quad 2\dot{x}(0) = 0 \ , \ 2\dot{x}(1) = -2x(1)$$

or

$$\ddot{x} = -1/2 \, ; \quad \dot{x}(0) = 0 \ , \ \dot{x}(1) = -x(1) \, . \tag{4.4}$$

The general solution of the Euler-Lagrange equation $\ddot{x} = -1/2$ is

$$x = -\frac{1}{4}\, t^2 + C_1 t + C_2 \ .$$

Note that

$$\dot{x} = -\frac{1}{2}\, t + C_1 \ .$$

The boundary conditions imply

$$0 = \dot{x}(0) = C_1 \ , \quad \dot{x}(1) = -\frac{1}{2} + C_1 = -\left(-\frac{1}{4} + C_1 + C_2\right) = -x(1) \ .$$

Hence, $C_1 = 0$, $C_2 = 3/4$. Thus, there is a unique extremal

$$\hat{x}(t) = -\frac{1}{4}\, t^2 + \frac{3}{4} \ .$$

One can prove that $\hat{x}(t) \in \text{absmin}\ Pr$. The minimal value

$$\mathcal{B}_{\text{ min}} = \mathcal{B}\left[\hat{x}(t)\right] = -\frac{1}{3} \ .$$

Computations in MAPLE.

```
> restart;
> ode:=2*diff(x(t),t$2)+1=0;
```

$$ode := 2\,(\tfrac{\partial^2}{\partial t^2}\,\mathrm{x}(t)) + 1 = 0$$

```
> BC:=D(x)(0)=0, D(x)(1)=-x(1);  # Boundary conditions
```

$$BC := \mathrm{D}(x)(0) = 0,\ \mathrm{D}(x)(1) = -\mathrm{x}(1)$$

```
> dsolve({ode,BC});
```

$$\mathrm{x}(t) = -\frac{1}{4}\,t^2 + \frac{3}{4}$$

```
> xx:=t-> -1/4*t^2+3/4; # Re-definition of solution for the next
```

$$xx := t \to -\frac{1}{4}\,t^2 + \frac{3}{4}$$

```
> B:=Int((diff(x(t),t))^2-x(t),t=0..1)+(x(1))^2;  # symbolic form
```

$$B := \int_0^1 (\tfrac{\partial}{\partial t}\,\mathrm{x}(t))^2 - \mathrm{x}(t)\,dt + \mathrm{x}(1)^2$$

```
> BB:=int((diff(xx(t),t))^2-xx(t),t=0..1)+(xx(1))^2; #The extremal value:
```

$$BB := \frac{-1}{3}$$

4.3. Isoperimetric problem

Statement of the isoperimetric problem with fixed endpoins *Pr3.*

In the space $C^{(1)}[t_0,\, t_1]$ of all continuously differentiable functions $x = x(t)$, $t \in [t_0,\, t_1]$, to find the functions such that

1. *function $x = x(t)$ minimizes or maximizes the value of the functional $\mathcal{J}_0$:*

$$\mathcal{J}_0[x(t)] = \int_{t_0}^{t_1} f_0(t, x(t), \dot{x}(t))\, dt\ \to \mathrm{extr}\,,$$

where f_0 is a given function (of class $C^{(1)}$) on three variables, provided

2.
$$\mathcal{J}_i[x(t)] = \int_{t_0}^{t_1} f_i(t, x(t), \dot{x}(t))\, dt = A_i\ , \tag{4.5}$$

where f_i, $i = 1\,,\, \ldots,\, m$, are given functions (of class $C^{(1)}$) on three variables, A_i, $i = 1\,,\, \ldots,\, m$, are given numbers, and

3. *$x(t_0) = x_0$, $x(t_1) = x_1$ (boundary conditions).*

We will refer to this problem as *Pr3*. In this problem the functions $f_0,\, \ldots\,,\, f_m$ are called *integrants*, the constraints (4.5) are called *isoperimetric*. The functions $x(t) \in C^{(1)}[t_0,\, t_1]$ which satisfy the constraints (4.5) and the boundary conditions are called *admissible.*

Definition 5. We say that the admissible function $\hat{x}(t)$ provides a (weak) local minimum to $\mathcal{J}_0$ in the problem *Pr3* if the inequality

$$\mathcal{J}_0[\hat{x}(t)] \leqslant \mathcal{J}_0[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we write

$$\hat{x}(t) \in \text{locmin } \mathit{Pr3}\,.$$

Definition 6. We say that the admissible function $\hat{x}(t)$ provides a (weak) local maximum to $\mathcal{J}_0$ in the problem *Pr3* if the inequality

$$\mathcal{J}_0[\hat{x}(t)] \geqslant \mathcal{J}_0[x(t)]$$

is satisfied for all admissible functions $x(t)$ from a certain neighborhood of the function $\hat{x}(t)$ in the space $C^{(1)}[t_0,\, t_1]$. In this case we will write

$$\hat{x}(t) \in \text{locmax } \mathit{Pr3}\,.$$

We use also the other notions and notation introduced for *Pr1*.

Necessary conditions for extremals in the isoperimetric problem.

One can prove the following theorem which gives necessary conditions for a function $\hat{x}(t)$ to be an extremal for $\mathcal{J}_0$.

Theorem 3. Let $\hat{x}(t)$ provide a (weak) local extremum in *Pr3*. Suppose that all the integrants f_i are functions of class $C^{(1)}$ in some neighborhood of the set $\{(t, \hat{x}, \dot{\hat{x}})\,|\, t \in [t_0, t_1]\}$. Then there exist numbers $\lambda_0, \ldots, \lambda_m$ such that at least one of them is nontrivial and such that for the lagrangian

$$L = L(t, x, \dot{x}) = \sum_{i=0}^{m} \lambda_i f_i(t, x, \dot{x})$$

the Euler-Lagrange equation

$$\frac{d}{dt}\hat{L}_{\dot{x}} = \hat{L}_x$$

is satisfied.

Remark. The numbers $\lambda_0, \ldots, \lambda_m$ are called *Lagrange multipliers*. It follows that for finding extremals we need to solve the boundary value problem for the differential equation (4.1) with the lagrangian L and to find if necessary the Lagrange multipliers. A vector $(\lambda_0, \ldots, \lambda_m) \neq 0$ can be considered up to the scaling by a nontrivial factor.

Example 4.4. Solve the following problem.

$$\mathcal{J}_0[x(t)] = \int_0^1 \dot{x}^2\, dt \;\to \text{extr}\;; \qquad \int_0^1 x\, dt = 0\,, \quad x(0) = 0\,,\; x(1) = 1\,.$$

The lagrangian for this problem is

$$L = \lambda_0\, \dot{x}^2 + \lambda_1\, x\,.$$

It follows that

$$\frac{\partial L}{\partial x} = \lambda_1\,, \quad \frac{\partial L}{\partial \dot{x}} = 2\lambda_0\, \dot{x}\;.$$

The Euler-Lagrange equation

$$\frac{d}{dt}(2\lambda_0\, \dot{x}) = \lambda$$

or

$$2\lambda_0\,\ddot{x} = \lambda_1 \qquad (4.6)$$

provides a necessary condition for extremals.

If $\lambda_0 = 0$ in (4.6) then $\lambda_1 = 0$, too, and there is no extremals at all in this case. Otherwise, let $2\lambda_0 = 1$. The equation (4.6) turns into

$$\ddot{x} = \lambda_1 \ . \qquad (4.7)$$

The general solution of the equation (4.7) is

$$x = \frac{1}{2}\lambda_1 t^2 + C_1 t + C_2 \ .$$

The boundary condition $x(0) = 0$ implies

$$C_2 = 0 \ .$$

The boundary condition $x(1) = 1$ implies

$$\frac{1}{2}\lambda_1 + C_1 = 1 \qquad (C_2 = 0).$$

It follows from the isoperimetric constraint that

$$0 = \int_0^1 x\,dt = \int_0^1 \left(\frac{1}{2}\,\lambda_1 t^2 + C_1 t\right) dt = \frac{1}{6}\,\lambda_1 + \frac{1}{2}\,C_1 \ .$$

The system

$$\frac{1}{2}\lambda_1 + C_1 = 1 \ , \qquad \frac{1}{6}\,\lambda_1 + \frac{1}{2}\,C_1 = 0$$

has the solution $\lambda_1 = 6$, $C_1 = -2$.

Thus, there is a unique extremal

$$\hat{x}(t) = 3\,t^2 - 2t \ .$$

One can prove that $\hat{x}(t) \in \text{absmin}\ Pr$. The minimal value

$$\mathcal{J}_{0,\,\min} = \mathcal{J}_0[\hat{x}(t)] = 4 \ .$$

Computations in MAPLE.

```
> restart;
> ode:= diff(x(t),t$2) = lambda;
```

$$ode := \frac{\partial^2}{\partial t^2}\,\mathrm{x}(t) = \lambda$$

```
> BC :=        x(0)=0, x(1)=1;   #Boundary conditions
```

$$BC := \mathrm{x}(0) = 0, \mathrm{x}(1) = 1$$

```
> dsolve(ode );
```

$$\mathrm{x}(t) = \frac{1}{2}\,\lambda\,t^2 + \_C1\ t + \_C2$$

```
> dsolve({ode,BC} );
```

$$x(t) = \frac{1}{2}\,\lambda\,t^2 + (-\frac{1}{2}\,\lambda + 1)\,t$$

```
> x:=(t,lambda)->1/2*lambda*t^2+(-1/2*lambda+1)*t;
```

$$x := (t,\,\lambda) \to \frac{1}{2}\,\lambda\,t^2 + (-\frac{1}{2}\,\lambda + 1)\,t$$

```
> lambda_1:=solve(int(x(t,lambda),t=0..1)=0,lambda); #isoperimetric
```

$$lambda\_1 := 6$$

```
> x(t,lambda_1);
```

$$3\,t^2 - 2\,t$$

```
> J:=int((diff(x(t,lambda_1),t)^2),t=0..1);
```

$$J := 4$$